

Recall:

$$\frac{d}{dx}[y] = \frac{dy}{dx} \text{ since the variables are different}$$

Similarly:

$\frac{d}{dt}[y] = \frac{dy}{dt}$ and $\frac{d}{dt}[x] = \frac{dx}{dt}$ since the variables don't match we need to use the chain rule. The "t" refers to time, so how does y change with respect to time, how does x change with respect to time?

Example 1: Finding Related Rates

- a. Suppose x and y are both differentiable functions of t and are RELATED by the equation

$y = x^2 + 3$. Find $\frac{dy}{dt}$ when $x = 1$, given that $\frac{dx}{dt} = 2$ when $x = 1$

Equation:

$$y = x^2 + 3$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 + 3]$$

Given rates and/or values:

$$x = 1 \quad \frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(1)(2)$$

Find:

$$\frac{dy}{dt}$$

$$\frac{dy}{dt} = 4$$

- b. Suppose x and y are both differentiable functions of t and are RELATED by the equation

$y = x^2 - 3x$. Find $\frac{dy}{dt}$ when $x = 3$, given that $\frac{dx}{dt} = 2$ when $x = 3$

Equation:

$$y = x^2 - 3x$$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2 - 3x]$$

Given rates and/or values:

$$x = 3 \quad \frac{dx}{dt} = 2$$

$$\frac{dy}{dt} = 2x \frac{dx}{dt} - 3 \frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(3)(2) - 3(2) = 12 - 6$$

Find:

$$\frac{dy}{dt}$$

$$\frac{dy}{dt} = 6$$

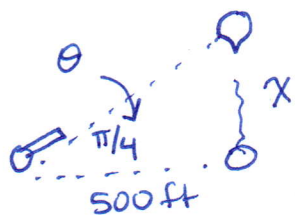
Guidelines for Solving Related-Rate Problems

1. **Understand the problem.** Identify all given quantities and quantities to be determined. Make a sketch and label the quantities.
2. **Develop a mathematical model.** DRAW A PICTURE!!! (most often these problems involve geometric figures) Label the parts that are important, distinguish between constant values and variable values. (e.g. at $x = 2$, $\frac{dx}{dt} = 7$ means ONLY when $x = 2$ is this the value for $\frac{dx}{dt}$)
3. **Write an equation relating the variables** whose rates of change are either given or to be determined. Often this is a geometric formula.
4. **Differentiate both sides of the equation WITH RESPECT TO TIME.** Be sure to follow all differentiation rules, chain, product, quotient, etc.
5. **Substitute values for any quantities that depend on time.** Substitute into the resulting equation all known values for the variables and their rates of change. ALWAYS TAKE THE DERIVATIVE THEN SUBSTITUTE. Then solve for the required rate of change.
6. **Interpret the solution.** Translate your mathematical result into the problem setting and decide whether the result makes sense.

Verbal Statement	Mathematical Model
A spherical balloon is inflated with gas at the rate of 500 cubic cm per minute.	$\frac{dV}{dt} = 500 \text{ cm}^3/\text{min}$
All edges of a cube are expanding at a rate of 3 cm per second.	$\frac{ds}{dt} = 3 \text{ cm/sec}$
At a sand and gravel plant, sand is falling onto a conical pile at a rate of 10 cubic feet per minute.	$\frac{dV}{dt} = 10 \text{ ft}^3/\text{min}$
A crankshaft rotates counterclockwise at a constant rate of 200 revolutions per minute.	$\frac{d\theta}{dt} = 200 \text{ rev/min}$
A player running from 2 nd to 3 rd base at a speed of 28 ft per second is 30 ft from third base.	$\frac{dx}{dt} = 28 \text{ ft/sec}$ $x = 30 \text{ ft}$

- Changing at a "constant rate" means $\frac{d\blacksquare}{dt}$ is the same no matter what the value of \blacksquare is.
- Changing at a "variable rate" means $\frac{d\blacksquare}{dt}$ depends on the value of \blacksquare .

Example 2: A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the balloon from the lift off point. At the moment the range finder's elevation angle is $\frac{\pi}{4}$, the angle is increasing at the rate of .14 radians per minute. How fast is the balloon rising at that moment?



$$\tan \theta = \frac{x}{500}$$

$$\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{500} \cdot \frac{dx}{dt}$$

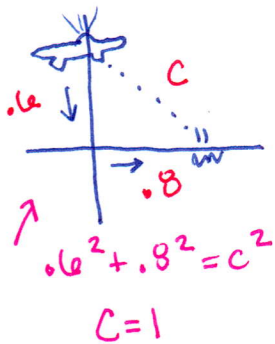
$$500 \cdot \sec^2 \frac{\pi}{4} \cdot (.14) = \frac{dx}{dt}$$

$$\frac{dx}{dt} = 140 \text{ ft/min}$$

The balloon is rising at a rate of 140 ft/min when the angle of elevation is $\pi/4$.

Speed = |velocity|

Example 3: A police cruiser, approaching a right angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 miles north of the intersection and the car is 0.8 miles to the east. The police determines with radar that the distance between them is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?



$$x^2 + y^2 = C^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2C \frac{dC}{dt}$$

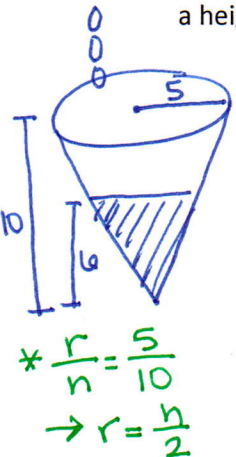
$$0.8 \left(\frac{dx}{dt} \right) + 0.6 (-60) = 1(20)$$

$$\frac{dx}{dt} = \frac{20 + 36}{0.8} = 56 \div \frac{4}{5} = 56 \cdot \frac{5}{4} = 70$$

$\frac{dy}{dt}$ getting closer

The speed of the car is 70 mph when the cop measures.

Example 4: Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?



$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{3} \pi \left(\frac{n}{2} \right)^2 h$$

$$V = \frac{\pi}{12} h^3$$

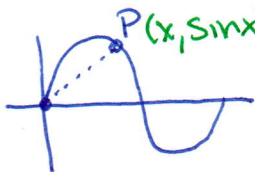
$$\frac{dV}{dt} = \frac{3\pi}{12} h^2 \cdot \frac{dh}{dt}$$

$$9 = \frac{\pi}{4} (6)^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9 \cdot 4}{36 \cdot \pi} = \frac{1}{\pi} \approx .32$$

The water level is rising at a rate of .32 ft/min when the water is 6 ft deep.

Example 5: Find the rate of change of the distance between the origin and a moving point on the graph of $y = \sin x$ if $\frac{dx}{dt} = 2$ cm per sec. and $x = \pi/4$.



$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$D = \sqrt{(0 - x)^2 + (0 - \sin x)^2}$$

$$D^2 = x^2 + \sin^2 x \rightarrow D = \sqrt{x^2 + \sin^2 x}$$

$$2D \cdot \frac{dD}{dt} = 2x \frac{dx}{dt} + 2 \sin x \cdot \cos x \cdot \frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{2(x) \frac{dx}{dt} + 2 \sin x \cos x \frac{dx}{dt}}{2D}$$

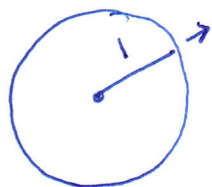
$$\frac{dD}{dt} = \frac{\left(\frac{\pi}{2} \right) (2) + \sin \left(\frac{\pi}{4} \right) \cos \left(\frac{\pi}{4} \right) \cdot 2}{\sqrt{\left(\frac{\pi}{4} \right)^2 + \left(\sin \frac{\pi}{4} \right)^2}} = \frac{\pi + \left(\frac{\sqrt{2}}{2} \right) \left(\frac{\sqrt{2}}{2} \right) \cdot 2}{\sqrt{\frac{\pi^2}{16} + \frac{1}{2}}} = \frac{\pi + 1}{\sqrt{\frac{\pi^2}{16} + \frac{1}{2}}} \approx 3.91$$

Example 6: A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius of the outer ripple is increasing at a rate of 1 ft/sec. When the radius is 4 ft, at what rate is the total area A of the disturbed water changing?

$$\frac{dA}{dt}$$

$$\frac{dr}{dt}$$

$$r$$



$$A \rightarrow \text{ft}^2$$

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi(4)(1)$$

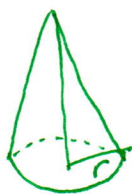
$$\frac{dA}{dt} = 8\pi$$

The rate at which the total area of the disturbed water is changing when the radius is 4 ft is $8\pi \text{ ft}^2/\text{sec}$.

Example 7: The formula for volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rate of change of the volume if the radius is changing at a rate of 2 inches per minute and the height is three times the radius when (a) radius is 6 inches, and (b) the radius is 24 inches.

$$\frac{dr}{dt}$$

$$\frac{dV}{dt}$$



$$h = 3r$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2 (3r)$$

$$V = \pi r^3$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

$$h = 3r$$

$$a) \frac{dV}{dt} = 3\pi(6)^2(2) = 216\pi$$

The volume is changing at a rate of $216\pi \text{ in}^3/\text{min}$ when the radius is 6 in.

$$b) \frac{dV}{dt} = 3\pi(24)^2(2) = 3456\pi$$

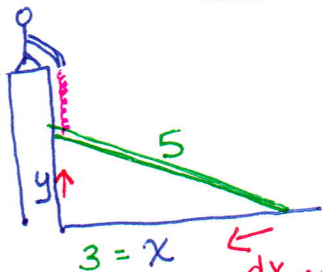
The volume is changing at a rate of $3456\pi \text{ in}^3/\text{min}$.

Example 8: A construction worker pulls a 5 m plank up the side of a building by means of a rope tied to one end of the plank. Assume the opposite end of the plank follows a path perpendicular to the wall of the building and the worker pulls the rope up at a rate of .15 meters per second. How fast is the end of the plank sliding along the ground when the base of the ladder is 3 meters from the wall?

← constant

$$\frac{dy}{dt}$$

$$\frac{dx}{dt}$$



$$3 = x$$

$\frac{dx}{dt}$ neg ble dist. is getting smaller

$$x^2 + y^2 + 5^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$(3) \frac{dx}{dt} + (4)(.15) = 0$$

$$3 \cdot \frac{dx}{dt} = -.6$$

$$\frac{dx}{dt} = \frac{-.6}{3} = -.2$$

When the base of the plank is 3 m from the wall the plank is sliding along the ground at a rate of -0.2 m/s .

$$3^2 + y^2 = 5^2$$

$$y = 4$$