27. Integration w/ u-substitution

Example 1: $\int \underline{x} \cdot (\underbrace{x^2+1}_{u})^2 \underline{dx}$

① Choose "u" from inside

$u = x^2+1$

$du = 2\underbrace{x \cdot dx}$

$\underbrace{x \cdot dx}_{\text{goal } x\text{'s}} = \underbrace{\tfrac{1}{2} du}_{u\text{'s + \#'s}} = \frac{du}{2}$

② replace all x's w/ u's

$\frac{1}{2}\int u^2 \, du$

③ integrate w/ smaller u

$= \frac{1}{2} \cdot \frac{u^3}{3} + C = \boxed{\frac{(x^2+1)^3}{6} + C}$

④ replace u w/ x's

Example 1a: $\int_0^2 x(x^2+1)^2 dx$

Change of Bounds

when $x=0$, $u = \overset{x^2+1}{0^2+1} = 1$

when $x=2$, $u = 2^2+1 = 5$

$= \frac{1}{2}\int_1^5 u^2 du = \frac{1}{2}\left(\frac{u^3}{3}\right]_1^5 = \frac{1}{2}\left(\frac{5^3}{3} - \frac{1}{3}\right)$

$= \frac{1}{2}\left(\frac{125}{3} - \frac{1}{3}\right) = \frac{1}{\cancel{2}}\left(\frac{\cancel{124}}{3}\right) = \boxed{\frac{62}{3}}$

Example 2: $\int \underline{x} \sqrt{\underbrace{x^2+1}_{u}} \, \underline{dx}$

$u = x^2+1$

$du = 2\underbrace{x \cdot dx}$

$x \cdot dx = \frac{1}{2} du$

$= \frac{1}{2}\int u^{1/2} du$

$= \frac{1}{2} \frac{u^{3/2}}{3/2} + C = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C$

Example 2a: $\int_4^5 x\sqrt{x^2+1} \, dx$

when $x=4$, $u = \overset{x^2+1}{17}$

when $x=5$, $u = 26$

$= \frac{1}{2}\int_{17}^{26} u^{1/2} du$

Example 3: $\int \sin x \sqrt{\underbrace{\cos x}_{u}}\, dx$

$u = \cos x$

$du = -\underbrace{\sin x\, dx}_{\text{isolate what I want to replace}}$

$\sin x \cdot dx = -du$

$-\int u^{1/2}\, du$

$= \frac{-2u^{3/2}}{3} + C = \frac{-2(\cos x)^{3/2}}{3} + C$

Example 3a: $\int_0^{\pi/4} \sin x \sqrt{\cos x}\, dx$

when $x = 0$, $u = \cos(0) = 1$

when $x = \pi/4$, $u = \cos(\pi/4) = \sqrt{2}/2$

$= -\int_1^{\sqrt{2}/2} u^{1/2}\, du = +\int_{\sqrt{2}/2}^{1} u^{1/2}\, du$

Example 4: $\int \cos x \sqrt{\underbrace{1-\cos^2 x}_{u}}\, dx$

$u = 1 - \cos^2 x$

$du = +2\cos x \cdot \sin x\, dx$

$\cos x\, dx = \frac{du}{2 \cdot \sin x}$

$= \frac{1}{2}\int u^{1/2} \cdot \frac{du}{\sin x}$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\int \cos x \sqrt{\underbrace{1-\cos^2 x}_{u}}\, dx$

$u = 1 - \cos^2 x = \sin^2 x$

$du = 2\sin x \cdot \cos x\, dx$

$\cos x\, dx = \frac{du}{2\sin x}$

$\int \cos x \sqrt{\sin^2 x}\, dx$

$\sqrt{u^2} = u$

$= \int \cos x \cdot \sin x\, dx$

$u = \cos x$

$du = -\sin x\, dx$

$u = \sin x$

$du = \cos x\, dx$

$-\int u\, du$

$\int u\, du$

$-\frac{u^2}{2} + C = \boxed{\frac{-\cos^2 x}{2} + C}$

$\frac{u^2}{2} + C = \boxed{\frac{\sin^2 x}{2} + C}$

These have the SAME derivative!

Example 5: $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

try $u = x$ $\quad du = dx \quad \rightarrow \int \frac{\sin\sqrt{u}}{\sqrt{u}}\,du$ $\quad$ no change, bad u!

try $u = \sin\sqrt{x}$ $\quad du = \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,dx$

$$\rightarrow \frac{dx}{\sqrt{x}} = \underbrace{\frac{2\,du}{\cos\sqrt{x}}}_{\text{not just } u\text{, bad } u!}$$

try $u = \sqrt{x}$ $\quad du = \frac{1}{2\sqrt{x}}\,dx$

$$\rightarrow \underbrace{\frac{dx}{\sqrt{x}}}_{\text{just } x\text{'s}} = \underbrace{2\,du}_{\text{just } u + \#} \quad \rightarrow 2\int \sin u\,du$$

$$= -2\cos u + C$$

$$= \boxed{-2\cos\sqrt{x} + C}$$