

8 The Chain Rule

Investigation

$$P(x) = (x+3)^3 = (x+3)(x+3)$$

$$(x+3) \quad \cancel{\times} \quad (x+3)$$

1 1

$$\begin{aligned} P'(x) &= 1(x+3) + 1(x+3) \\ &= 2(x+3) \end{aligned}$$

$$\begin{aligned} P(x) &= (x-7)^3 \\ &= (x-7)(x-7)(x-7) \end{aligned}$$

$$(x-7) \quad \cancel{\times} \quad (x-7)^2$$

1 2(x-7)

$$\begin{aligned} P'(x) &= 1(x-7)^2 + 2(x-7)^2 \\ &= 3(x-7)^2 \end{aligned}$$

$$\left\{ P'(x) = 3(x-7)^2 \right.$$

$$\frac{d}{dx}(x-7)^5 = 5(x-7)^4$$

$$Q(x) = (2x-3)^2 = (2x-3)(2x-3)$$

$$\begin{matrix} 2x-3 & & 2x-3 \\ & 2 & & 2 \end{matrix}$$

- ① guess $Q'(x)$
② use $\cancel{\times}$ find $Q'(x)$
 $Q'(x) = ?(2x-3)$

$$Q'(x) = \underbrace{(2)(2x-3)}_{\text{exponent } u} + (2)(2x-3)$$

$$2 \left[\underbrace{(2)}_{\text{exponent } u} \underbrace{(2x-3)}_u \right]$$

$$Q(x) = (6x+1)^3$$

~~$(6x+1)^3 = 6 \cdot (6x+1)^2$~~
 ~~6~~ ~~$2(6x+1) \cdot 6$~~

$$Q'(x) = \underset{\text{exp}}{\overset{?}{3}} \cdot \underset{u'}{6} \cdot \underset{u}{\underbrace{(6x+1)}^2}$$

$$Q'(x) = \underbrace{2(6x+1)^2}_{u'} + 2 \cdot \underbrace{6(6x+1)^2}_{u'}$$

$u = 6x+1$

$$Q'(x) = \underset{\text{exp}}{\overset{?}{3}} \cdot \underset{u'}{6} \cdot \underset{u}{\underbrace{(6x+1)}^2}$$

Chain Rule

$$f(x) = u^n \quad \text{where } u \text{ is a fn of } x$$

(not just an x , many of them)

Power chain rule: $f'(x) = n \cdot u^{n-1} \cdot u'$

Example 1: Use the chain rule to find each derivative

A) $f(x) = (x^2 - 2x + 1)^5$

$$u = x^2 - 2x + 1$$

$$u' = 2x - 2$$

$$f'(x) = 5(x^2 - 2x + 1)^4 \cdot (2x - 2)$$

B) $g(x) = (\sin x + 2x^2 + \log_3 x)^{17}$

$$u = \sin x + 2x^2 + \log_3 x$$

$$u' = \cos x + 4x + \frac{1}{x \ln 3}$$

$$g'(x) = 17(\sin x + 2x^2 + \log_3 x)^{16} \cdot (\cos x + 4x + \frac{1}{x \ln 3})$$

C) $h(x) = 3(2x^7 - 5x + \tan x)^{12}$

$$u = 2x^7 - 5x + \tan x$$

$$u' = 14x^6 - 5 + \sec^2 x$$

$$h'(x) = 3 \cdot 12(2x^7 - 5x + \tan x)^{11} \cdot (14x^6 - 5 + \sec^2 x)$$

Trig : P Power T trig A angle

$$P(x) = \sin (\underbrace{3x^2 - 7}_{\text{Angle}})$$

$$u = 3x^2 - 7$$

$$u' = 6x$$

Outside = "sin"

$$P'(x) = \cos(3x^2 - 7) \cdot 6x$$

$$B(x) = \tan (8x^3 - 5x + 3)$$

$$u = 8x^3 - 5x + 3$$

$$u' = 24x^2 - 5$$

$$B'(x) = \underbrace{\sec^2(8x^3 - 5x + 3)}_{\text{outside}} \cdot \underbrace{(24x^2 - 5)}_{\text{inside}}$$

Outside = "tan"

$$R(x) = \sin^3(3x^2) = \underbrace{\sin(3x^2)}_V \cdot \underbrace{\sin(3x^2)}_V \cdot \underbrace{\sin(3x^2)}_V$$

$$u = 3x^2 \quad \text{out} = \text{"sin"}$$

$$u' = 6x$$

$$\frac{d}{dx} [\sin(3x^2)] = \cos(3x^2) \cdot 6x$$

$$V = \sin(3x^2)$$

$$V' = \cos(3x^2) \cdot 6x$$

$$\text{out} = ()^3$$

$$Q(x) = \tan^4(6x - 1)$$

$$\begin{array}{lcl} P: 4 & \rightarrow & 4(\ ? \)^3 \\ T: \tan & \rightarrow & \sec^2(\ ?) \\ A: 6x-1 & \rightarrow & 6 \end{array}$$

$$Q'(x) = [4(\tan(6x-1))^3 \cdot \sec^2(6x-1) \cdot 6]$$

$$W(x) = \cos^7(4x^2 + \ln x)$$

$$W'(x) = 7(\cos(4x^2 + \ln x))^6 \cdot \sin(4x^2 + \ln x) \cdot (8x^2 + \frac{1}{x})$$

$$P: ()^7 \rightarrow 7()^6$$

$$T: \cos() \rightarrow -\sin()$$

$$A: 4x^2 + \ln x \rightarrow 8x^2 + \frac{1}{x}$$



$$Q(x) = \tan^4(6x-1)$$

Power: $(\underline{\quad})^4 \rightarrow 4(\underline{\quad})^3$

Trig: $\tan(\underline{\quad}) \rightarrow \sec^2(\underline{\quad})$

Angle: $(6x-1) \rightarrow (6\underline{\quad})$

$$\begin{aligned} Q'(x) &= 4(\tan(6x-1))^3 \cdot \sec^2(6x-1) \cdot 6 \\ &= 4\tan^3(6x-1) \cdot \sec^2(6x-1) \cdot 6 \\ &\rightarrow 24 \tan^3(6x-1) \cdot \sec^2(6x-1) \\ &= 24 \sec^2(6x-1) \tan^3(6x-1) \rightarrow \frac{24}{\cos^5(6x-1)} \cdot \sin^3(6x-1) \end{aligned}$$

Example 2: Using the Chain Rule

A) Find the equation of the NORMAL line to
 $y = (2x-7)^3$ at $x=2$

$$\begin{aligned} \tan &= \frac{a}{b} \\ \text{nor} &= -\frac{b}{a} \end{aligned}$$

$m = \text{deriv}$ (slope of tan line)
 (x, y)

1st find (x, y)
 $(2, -27)$

$$y(2) = (2(2)-7)^3 = -27$$

$$\begin{aligned} y &= u^3 \\ y' &= 3u^2 \cdot u' \end{aligned}$$

2nd find $m = y'$ at $x=2$

$$y' = 3(\underbrace{2x-7}_u)^2 \cdot \underbrace{2}_{u'}$$

$$\begin{aligned} y'(2) &= 6(2(2)-7)^2 \\ &= 6(-3)^2 \end{aligned}$$

$= 54 \leftarrow$ slope of tangent to y at $x=2$

3rd find normal slope

$$\begin{aligned} m_{\tan} &= \frac{a}{b} \rightarrow m_{\text{nor}} = -\frac{b}{a} \\ m_{\tan} &= 54 \rightarrow m_{\text{nor}} = -\frac{1}{54} \end{aligned}$$

4th Write eqn: $(2, -27) \quad m = -\frac{1}{54}$

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y + 27 &= -\frac{1}{54}(x - 2) \end{aligned}$$

B) Write the equation of the line normal to
 $y = \left(\frac{2}{x-1}\right)^3$ at $x=2$

$$(2, 8) \quad y' = 3 \left(\underbrace{\frac{2}{(x-1)}}_u \right)^2 \cdot \left(\underbrace{\frac{-2}{(x-1)^2}}_{u'} \right)$$

$$\begin{aligned} & 2 \nearrow \cancel{x-1} \\ & 0 \circlearrowleft \end{aligned}$$

$$\frac{(0)(x-1) - (1)(2)}{(x-1)^2}$$

$$\begin{aligned} y'(2) &= 3 \left(\frac{2}{1} \right)^2 \left(\frac{-2}{(1)^2} \right) \\ &= 3(4)(-2) \end{aligned}$$

$$y'(2) = -24 \quad \leftarrow \text{slope of } \underline{\text{tan}} \text{ line to } g \text{ at } x=2$$

$$m_{\text{nor}} = -\frac{1}{24}$$

$$\rightarrow y - 8 = -\frac{1}{24}(x - 2)$$

Ex 3: Shortcut for quotient Rule

$$y = \frac{2x-1}{3x^2-7} = (2x-1) \cdot \underbrace{(3x^2-7)^{-1}}_{\text{power}}$$

$$y = \frac{2}{(x-1)}$$

$$\begin{aligned} & 2x-1 \nearrow \cancel{x-1} \\ & 2 \circlearrowleft \end{aligned}$$

$$(3x^2-7)^{-1} - 1(3x^2-7)^{-2} \cdot (6x)$$

$$y = 2(x-1)^{-1}$$

$$2(3x^2-7)^{-1} + (-1)(3x^2-7)^{-2}(6x)(2x-1)$$

$$y' = 2(-1)(x-1)^{-2}$$

$$\frac{2(3x^2-7)}{(3x^2-7)(3x^2-7)} \quad \frac{(-1)(6x)(2x-1)}{(3x^2-7)^2}$$

$$= \frac{-2}{(x-1)^2}$$

$$\frac{2(3x^2-7) - (6x)(2x-1)}{(3x^2-7)^2}$$

Ex 4 | Use the chain rule

A) $y = \frac{6}{(3x^2-1)^4} = 6(3x^2-1)^{-4}$

$$y' = 6(-4)(3x^2-1)^{-5}(6x)$$

$$y' = \frac{6(-4)(6x)}{(3x^2-1)^5} = \frac{-144x}{(3x^2-1)^5}$$

$$\begin{array}{r} 2 \\ 36 \\ \times 4 \\ \hline 144 \end{array}$$

B) find y''

$$(()^5)^2 = ()^{10}$$

$$\begin{matrix} -144x & \cancel{\times} & (3x^2-1)^5 \\ -144 & \cancel{\times} & 5(3x^2-1)^4 \cdot 6x \end{matrix}$$

$$\begin{aligned} y'' &= \frac{(-144)(3x^2-1)^4(3x^2-1) - (30x)(3x^2-1)^3(-144x)}{(3x^2-1)^{10}} \\ &= \frac{(-144)(3x^2-1) - (30x)(-144x)}{(3x^2-1)^6} \end{aligned}$$

Ex5: Use the table to find each

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	$\frac{1}{3}$	-3
3	3	-4	2π	5

A) $2 \cdot f(x)$ want deriv at $x=2$
 $h(x) = 2 \cdot f(x)$ find $h'(2)$

$$\begin{aligned} h'(x) &= 2 \cdot f'(x) \\ h'(2) &= 2 \cdot f'(2) \\ &= 2 \cdot \left(\frac{1}{3}\right) = \boxed{\frac{2}{3}} \end{aligned}$$

B) $r(x) = 3f(x) + g^2(x)$ at $x=2$

$$(g(x))^2 = g(x) \cdot g(x) \quad \sin^2(x)$$

want $r'(2)$

$$\begin{aligned} r(x) &= 3f(x) + (g(x))^2 & g^2(x) = (3x+1)^2 & g(x) = 3x+1 \\ r'(x) &= 3 \cdot f'(x) + 2(g(x)) \cdot g'(x) \\ r'(2) &= 3 \cdot f'(2) + 2g(2) \cdot g'(2) \\ &= 3 \cdot \left(\frac{1}{3}\right) + 2(2)(-3) \\ &= 1 + -12 \\ &= -11 \end{aligned}$$

C) $d(x) = \frac{g(x)}{f^2(x)}$ $d'(3)$

$$d'(x) = \frac{g'(x) \cdot f^2(x) - 2 \cdot f(x) \cdot f'(x) \cdot g(x)}{f^4(x)}$$

$$= \frac{(5)(9) - 2(3)(2\pi)(-4)}{3^4} = \boxed{\frac{45 + 48\pi}{81}}$$

$$\begin{array}{c} g(x) \curvearrowleft \\ g'(x) \curvearrowleft \end{array} \begin{array}{c} (f(x))^2 \\ 2(f(x)) \cdot f'(x) \\ 3^{2\pi} \end{array}$$

$$\begin{array}{c} -4 \curvearrowleft \\ 5 \curvearrowleft \end{array} \begin{array}{c} 9 \\ 12\pi \end{array}$$

$$d'(x) = \frac{f(x)}{f(x)} \left[g'(x)f(x) - \frac{2}{f^3(x)} f'(x)g'(x) \right]$$

$$d'(3) = \frac{45 + 16\pi}{3} = \frac{3(15 + 16\pi)}{27}$$

D) $p(x) = \sqrt{f(x)}$, at $x=3$ $p'(3)$

p'

$$\frac{f(x)}{2\sqrt{f(x)}}$$