

9. Implicit Differentiation

Implicit

vs

Explicit

$$x^3 + y^3 - 9xy = 0$$

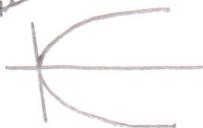


$$2y = x^2 + \sin y$$

$$x^2 + y^2 = 25$$



$$y^2 = x \rightarrow y = +\sqrt{x} \\ = -\sqrt{x}$$



$$y = 3x - 7$$

$$f(x) = 2x^2 - 4$$

↑ dependent variable
is isolated

(3, ↑)

value depends
on choice of x

$$3x - 2y = 7$$

↑ easily
solved for
y

$$y \rightarrow y' \rightarrow \frac{dy}{dx}$$

y'

$$\frac{d}{dx}[x^2] = 2x \cdot \frac{dx}{dx}$$

$$\frac{d}{dx}[(3x-1)^2] = 2(3x-1)' \cdot 3$$

$$\frac{d}{dx}[u^2] = 2u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[y^2] = 2y \cdot \frac{dy}{dx}$$

Implicit diff steps

- Differentiate BOTH Sides w.r.t. x
- Collect $\frac{dy}{dx}$ terms on same side, everything else goes to other side
- Factor out $\frac{dy}{dx} \rightarrow \frac{dy}{dx} \cdot (\dots) = \dots$
- Solve for $\frac{dy}{dx}$ by dividing.

Y Y

Ex 1] Use implicit diff to find $\frac{dy}{dx}$

A) $y^2 = x$

$$\frac{d}{dx}[y^2] = \frac{d}{dx}[x]$$

$$2y \cdot \frac{dy}{dx} = 1$$

$$2y \cancel{y} = 1$$

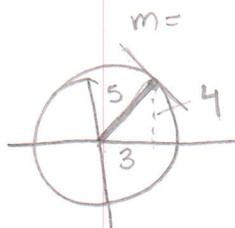
$$\frac{dy}{dx} = \frac{1}{2y}$$

B) $x^2 + y^2 = 25$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{2y \frac{dy}{dx}}{2y} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$



$$m_{\text{rad}} = 4/3$$
$$m_{\text{tan}} = -3/4$$

C) $2y = x^2 + \sin y$

$$2 \cdot \frac{dy}{dx} = 2x + \cos(y) \cdot \frac{dy}{dx}$$

$$2 \frac{dy}{dx} - \cos y \cdot \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} (2 - \cos y) = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2 - \cos y}$$

$$D) \quad x^3 + y^3 - 9xy = 0$$

$$\begin{array}{l} -9x \\ -9 \end{array} \quad \begin{array}{l} \nearrow \\ \searrow \end{array} \quad y \frac{dy}{dx}$$

$$3x^2 + 3y^2 \cdot \frac{dy}{dx} - 9y - 9x \frac{dy}{dx} = 0$$

$$3y^2 \frac{dy}{dx} - 9x \frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} (3y^2 - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x} = \frac{x^2 - 3y}{3x - y^2}$$

Example 2 Find the equation of the tangent and normal line for each. $m = \frac{dy}{dx}$ (,)

A) $x^2 - xy + y^2 = 7$ at $(-1, 2)$

$$2x - y - x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$x^2 - xy + y^2$
 $\frac{dy}{dx}$

$$\frac{dy}{dx} (2y - x) = y - 2x$$

$$2x - y = \frac{dy}{dx} (x - 2y)$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

~~$y - 2 = \frac{y - 2x}{2y - x} (x + 1)$~~ \leftarrow need to plug in $y - y_1 = m(x - x_1)$

$$\left. \frac{dy}{dx} \right|_{(-1, 2)} = \frac{(2) - 2(-1)}{2(2) - (-1)} = \frac{2 + 2}{4 + 1} = \frac{4}{5} \leftarrow \text{Slope of the tan line @ } (-1, 2)$$

Slope of normal line @ $(-1, 2) \Rightarrow -\frac{5}{4}$

tan line	$y - 2 = \frac{4}{5}(x + 1)$
normal	$y - 2 = -\frac{5}{4}(x + 1)$

Ex3 Finding higher order derivatives

A) $2x^3 - 3y^2 = 8$
 $6x^2 - 6y \frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{x^2}{y}$$

$$\frac{d}{dx} \left[\frac{d}{dx} \left[\frac{d}{dx} [y] \right] \right]$$

$$\frac{d^2 y}{dx^2} = \frac{(2xy - x^2 \cdot \frac{x^2}{y})}{(y^2)^2} \cdot \frac{y}{y}$$

$x^2 - xy + y^2$
 $\frac{dy}{dx} = \frac{x^2}{y}$

don't want $\frac{dy}{dx}$, want $x^2 y^2$

$$= \frac{2xy^2 - x^4}{y^3}$$

$$\frac{d^3 y}{dx^3}$$

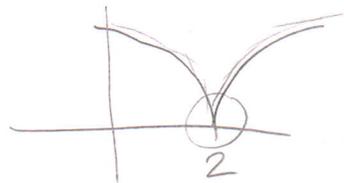
Fn not diff = 1) cusp / corner
 2) vert tan
 3) discontinuity

Ex 4] Algebraically finding when a derivative does not exist

A) $y = \sqrt{x}$
 $= x^{1/2}$
 $y' = \frac{1}{2} x^{-1/2}$
 $y' = \frac{1}{2\sqrt{x}}$

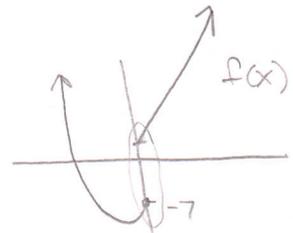
$x \neq 0$
 y is not differentiable at $x=0$

B) $y = \sqrt{|x-2|}$
 $y' = \frac{1}{2\sqrt{|x-2|}} \cdot \left(\frac{d}{dx}[|x-2|]\right)$



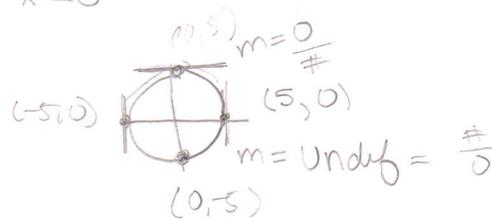
$\sqrt{|x-2|} \neq 0$
 $x \neq 2$

C) $f(x) = \begin{cases} x^2 + 3x - 7 & x \leq 0 \\ 4|x+1| & x > 0 \end{cases}$



$f'(x) = \begin{cases} 2x + 3 & x > 0 \\ 4 & x < 0 \end{cases}$ $\leftarrow x=0?$

D) $x^2 + y^2 = 25$
 $2x + 2y \cdot \frac{dy}{dx} = 0$
 $\frac{2y \frac{dy}{dx}}{2y} = -\frac{2x}{2y}$
 $\frac{dy}{dx} = -\frac{x}{y}$



horizontal tan when num = 0 $\begin{matrix} -x=0 \\ x=0 \end{matrix}$
 vert tan when den = 0 $y=0$

E) $y^2(2-x) = x^3$

Show $\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}$

$(2y) \left(\frac{dy}{dx} \right) (2-x) - y^2 = 3x^2$

$2y \cdot \frac{dy}{dx} \cdot (2-x) - y^2 = 3x^2$

$\frac{2y(2-x) \frac{dy}{dx}}{2y(2-x)} = \frac{3x^2 + y^2}{2y(2-x)}$

$\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}$

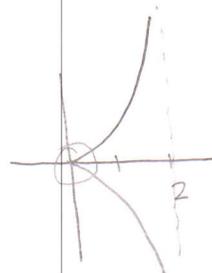
← when is tan line horizontal $3x^2 + y^2 = 0$

dy/dx does not exist when $x=2, y=0$
(denom = 0... bad)

need to check if $x=2, y=0$ are pts on the fn

$x=2 \quad y^2(2-2) = 2^3$
 $0 = 8$

$y=0 \quad 0^2(2-x) = x^3$
 $0 = x^3$
 $x=0$



at $(0,0)$ the derivative of $y^2(2-x) = x^3$ does not exist

AP style problems

MC → given imp relation $y^2(2-x) = x^3$

A
B
C
D
E
 $\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2-x)}$

→ given imp. relation find slope at $(,)$ → find $\frac{dy}{dx}$
→ plugin

→ given imp rel at what value of x or y does $\frac{dy}{dx} = \#$

→ given $\frac{dy}{dx}$, find $\frac{d^2y}{dx^2}$ (plugin $\frac{dy}{dx} =$)

→ where is tan line horizontal, vertical or not differentiable

FRQ: • given $y = -\ln(-x^3 + 3x^2 - 1)$
show $\frac{dy}{dx} = e^y(3x^2 - 6x)$

$$\underbrace{5}x^3$$

• given $\frac{dB}{dt} = \frac{1}{5}(100 - B)$ Find $\frac{d^2B}{dt^2}$

$$\begin{aligned}\frac{d^2B}{dt^2} &= \frac{1}{5}\left(0 - \frac{dB}{dt}\right) \\ &= \frac{1}{5}\left(-\left(\frac{1}{5}(100 - B)\right)\right) \\ &= -\frac{1}{5} \cdot \frac{1}{5}(100 - B)\end{aligned}$$

$$\frac{d^2B}{dt^2} = -\frac{1}{25}(100 - B)$$