

## 24 The Fundamental Theorem of Calculus

### Part 2

$$\int_a^b f(x) dx = F(b) - F(a)$$

Special notation  $= F(x) \Big|_a^b$  means  $\uparrow$

$$\begin{array}{c} \uparrow F(x) \\ f(x) \\ \downarrow f'(x) \end{array}$$

$$\int f'(x) dx = f(b) - f(a)$$

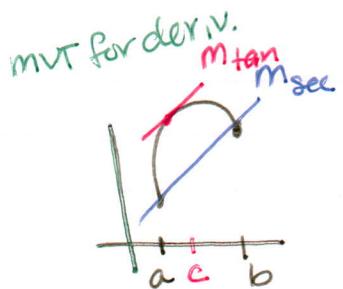
### Part 1

$$\frac{d}{dx} \int_a^x f(t) dt = f(x)$$

$$\int_a^b f(t) dt = F(b) - F(a)$$

$$\frac{d}{dx} \int_a^x f(t) dt = \frac{d}{dx} [F(x) - F(a)]$$

$$= f(x)$$

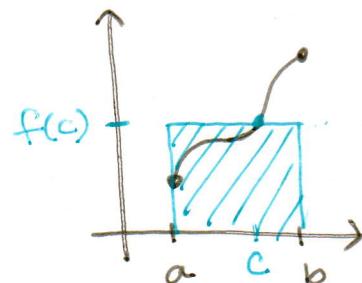


### Average Value Theorem / Mean Value Theorem for Integrals

If  $f$  is integrable on  $[a,b]$  then the average value of  $f$ , for some  $x$ -value  $c \in [a,b]$ , is

$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

$$(b-a) \cdot f(c) = \int_a^b f(x) dx$$



The mean value theorem for integrals gives the  $y$ -value where a single rectangle can be drawn to equal the area between  $f(x)$  + the  $x$ -axis

NOT MIDPOINT

$$S(t)$$

$$X'(t) = V(t)$$

$$F'(t) = f(t)$$

$$\text{ave velocity} = \frac{X(b) - X(a)}{b-a}$$

$\uparrow$   
 $V(t)$

$$f(c) = \frac{\int_a^b f(t) dt}{b-a}$$

$$f(c) = \frac{F(b) - F(a)}{b-a}$$

(FTC)

Example 1 Use the Fundamental Theorem of Calculus to evaluate the definite integral

$$A) \int_{-1}^3 (2x-2) dx = \left[ x^2 - 2x + C \right]_{-1}^3$$

$\overbrace{= (3+C) - (-1+C)}$   
 $= 0$

$$(3)^2 - 2(3) + C = 3 + C$$
$$(-1)^2 - 2(-1) + C = 3 + C$$

no need for  
" + C " because  
they will sum  
to 0

$$B) \int_1^2 (x^3 - 2x^2 + 1) dx = \left[ \frac{x^4}{4} - \frac{2x^3}{3} + x \right]_1^2$$
$$= \frac{16}{4} - \frac{16}{3} + 2 = 6 - \frac{16}{3}$$
$$= \frac{8}{12} - \frac{7}{12}$$
$$= \frac{1}{12}$$

$$\frac{18}{3} - \frac{16}{3} = \frac{2}{3} = \frac{8}{12}$$
$$\frac{1}{4} - \frac{2}{3} + 1$$
$$\frac{3}{12} - \frac{8}{12} + \frac{12}{12} = \frac{7}{12}$$

$$C) \int_{-1}^2 (-x^3 + x^2 - 2) dx = \left[ -\frac{x^4}{4} + \frac{x^3}{3} - 2x \right]_{-1}^2$$
$$= \left( -4 + \frac{8}{3} - 4 \right) - \left( \frac{-1}{4} - \frac{1}{3} + 2 \right)$$
$$= -10 + \frac{9}{3} + \frac{1}{4}$$
$$= -7 + \frac{1}{4}$$
$$= -\frac{28}{4} + \frac{1}{4} = -\frac{27}{4}$$

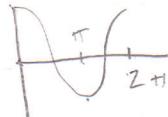
$$-\frac{24}{3} + \frac{8}{3} = -\frac{16}{3}$$
$$-\frac{3}{12} - \frac{4}{12} + \frac{12}{12} = \frac{7}{12}$$

$$D) \int_{-3}^0 -e^x dx = -e^x \Big|_{-3}^0 = -e^0 + e^{-3} = -1 + e^{-3}$$

$$= \frac{-e^3}{e^3} + \frac{1}{e^3}$$

$$= \frac{-e^3 + 1}{e^3} = \frac{1 - e^3}{e^3}$$

$$E) \int_0^\pi -2 \sin x dx = +2 (\cos x) \Big|_0^\pi$$



$$\begin{aligned} &= 2\cos\pi - 2\cos 0 \quad \text{or } 2(\cos\pi - \cos 0) \\ &= 2(-1) - 2(1) \quad = 2(-1 - 1) \\ &= -2 - 2 \quad = 2(-2) \\ &= -4 \quad = -4 \end{aligned}$$

$$F) \int_{-2}^{-1} 5x^{4/3} dx = \frac{5x^{4/3}}{4/3} \Big|_{-2}^{-1} = \frac{15}{4} \cdot x^{4/3} \Big|_{-2}^{-1}$$

~~$$\begin{aligned} &\frac{4}{3}\sqrt[3]{x^3} \\ &\sqrt[3]{(-1)^4} - \sqrt[3]{(-2)^4} \\ &(\sqrt[3]{-})^4 \end{aligned}$$~~

$$= \frac{15}{4} \left( \sqrt[3]{(-1)^4} - \sqrt[3]{(-2)^4} \right)$$

$$= \frac{15}{4} (1 - 2\sqrt[3]{2})$$

Example 2 Use the Fundamental Theorem of Calculus to find  $F'(x)$ .

A)  $F(x) = \int_{-7}^x -3(t-1)^{1/3} dt$

$$F'(x) = \frac{d}{dx} \int_{-7}^x -3(t-1)^{1/3} dt = \underbrace{-3(x-1)^{1/3}}_{f(x)}$$

B)  $F(x) = \int_{-1}^x (-t^2 + 2t) dt \quad F'(x) = -x^2 + 2x$

1 part FTC expanded

•  $\int_x^a f(t) dt = - \int_a^x f(t) dt$

$$\downarrow \qquad \qquad - \frac{d}{dx} \left[ \int_a^x f(t) dt \right] = -f(x) \qquad //$$

$$\frac{d}{dx} [F(a) - F(x)] = 0 - f(x) \qquad //$$

•  $\int_a^{g(x)} f(t) dt = F(g(x)) - F(a)$

$$\begin{aligned} \frac{d}{dx} \int_a^{g(x)} f(t) dt &= \frac{d}{dx} [F(g(x)) - F(a)] \\ &= f(g(x)) \cdot g'(x) \end{aligned}$$

C)  $F(x) = \int_x^2 e^t dt \quad F'(x) = -e^x$   
 $x$  is lower bound

D)  $F(x) = \int_4^{3x} e^t dt$

$$F'(x) = e^{3x} \cdot 3$$

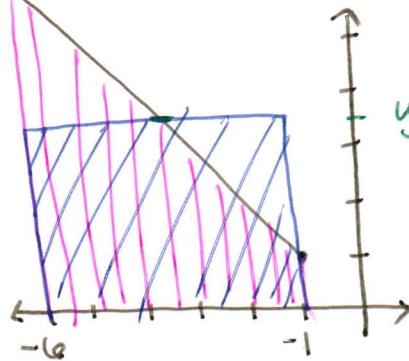
$$f(c) = \frac{1}{b-a} \int_a^b f(x) dx$$

Example 3 Find the average value of the function over the given interval.

A)  $f(x) = -x$   $[-6, -1]$

$$\frac{1}{-1 - (-6)} \int_{-6}^{-1} -x dx$$

$$= \frac{1}{5} \left( \frac{x^2}{2} \right) \Big|_{-6}^{-1} = \frac{1}{10} \left( 1 - 36 \right) = \frac{35}{10} = \frac{7}{2}$$



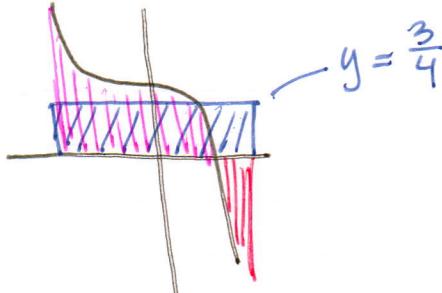
$y = 3.5$  ← y-value where I can draw a single rectangle to represent the area  
 $\int_{-6}^{-1} -x dx$

B)  $f(x) = -x^3 + x^2 + 1$   $[-1, 2]$   $\frac{3}{4}$

$$\frac{1}{2 - (-1)} \int_{-1}^2 -x^3 + x^2 + 1 dx = \frac{1}{3} \left( -\frac{x^4}{4} + \frac{x^3}{3} + x \right) \Big|_{-1}^2$$

$$= \frac{1}{3} \left[ \left( -\frac{16}{4} + \frac{8}{3} + 2 \right) - \left( -\frac{1}{4} - \frac{1}{3} - 1 \right) \right]$$

$$= \frac{1}{3} \left( \frac{9}{4} \right) = \frac{3}{4}$$



C)  $f(x) = x^2 + 4x - 2$   $[-4, 1]$

$$\frac{1}{1-(-4)} \int_{-4}^1 x^2 + 4x - 2 \, dx = \frac{-11}{3}$$

D)  $f(x) = x^2 + 4x$   $[-4, 1]$

$$\frac{1}{1-(-4)} \int_{-4}^1 x^2 + 4x \, dx$$

$$= \frac{1}{5} \left( \frac{x^3}{3} + 2x^2 \right) \Big|_{-4}^1$$

$$= \frac{1}{5} \left[ \left( \frac{1^3}{3} + 2(1)^2 \right) - \left( \frac{(-4)^3}{3} + 2(-4)^2 \right) \right]$$

$$= \frac{1}{5} \left[ \left( \frac{1}{3} + \frac{6}{3} \right) - \left( -\frac{64}{3} + \frac{32}{3} \right) \right]$$

$$= \frac{1}{5} \left( \frac{7}{3} - \frac{32}{3} \right)$$

$$= \frac{1}{5} \left( -\frac{25}{3} \right) = -\frac{5}{3}$$

$$-\frac{11}{3} + 2 =$$

$$-\frac{11}{3} + \frac{6}{3} = -\frac{5}{3}$$

- \* C is a vertical translation of D, so it makes sense that D is 2 units above the average value for C.