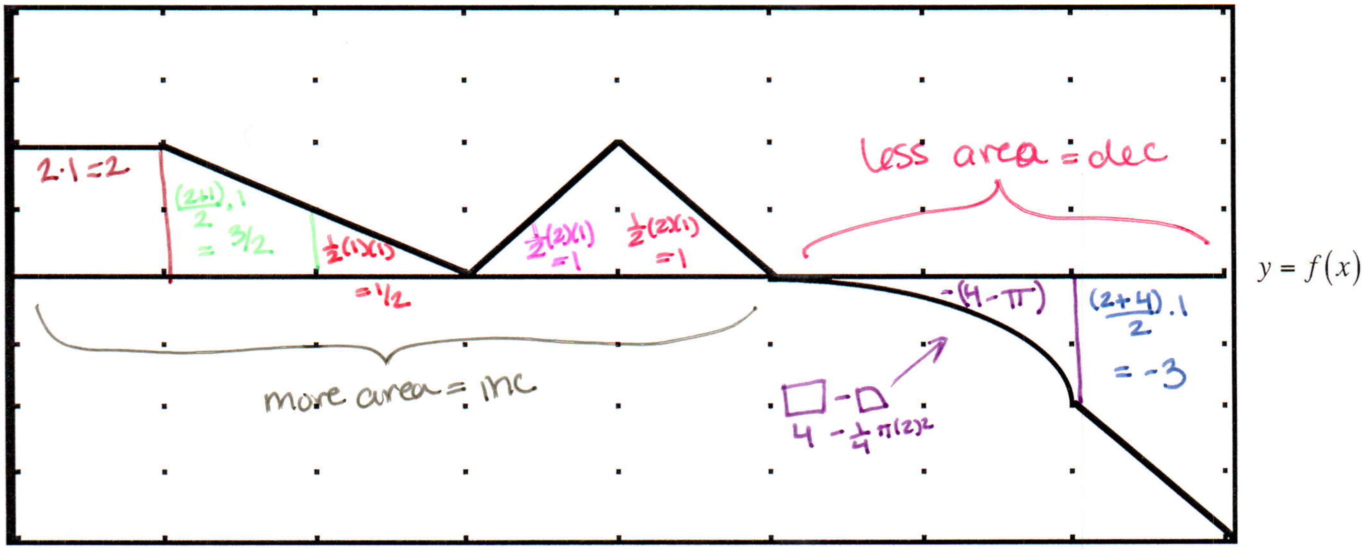
\# 25

The Accumulation Function - Homework



1. Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(x)$ is above (the graph consists of lines and a quarter circle)

F=area

$F' = f$ = y-values

a. Complete the chart

x	0	1	2	3	4	5	7	8
$F(x)$	0	2	$2+3/2 = \frac{7}{2}$	$\frac{7}{2}+\frac{1}{2}=4$	$4+1=5$	$5+1=6$	$6-(4-\pi)$ $=2+\pi$	$2+\pi-3$ $=-1+\pi$ >0
$F'(x)$	2	2	1	0	2	0	-2	-4

b. On what subintervals of [0, 8] is F increasing? Decreasing? Justify your answer.

F inc inc dec

F' + 3 + 5 −

Inc (0,5) $F' > 0$

Dec (5,8) $F' < 0$

c. Where in the interval [0, 8] does F achieve its minimum value? What is the minimum value? Justify answer.

min value occurs dec → inc or boundary pts ← lowest y-value

does not occur $\boxed{F(0)=0}$ $F(8) = -1+\pi > 0$

d. Where in the interval [0, 8] does F achieve its maximum value? What is the maximum value? Justify answer.

max value occurs inc → dec or boundary

↳ x = 5 $F(5) = 6$

e. Find the concavity of F and any inflection points. Justify answers.

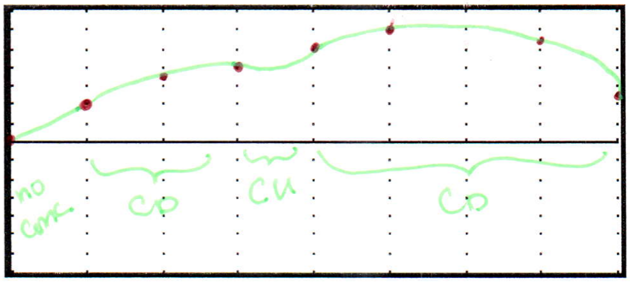
F no con. CD CU CD

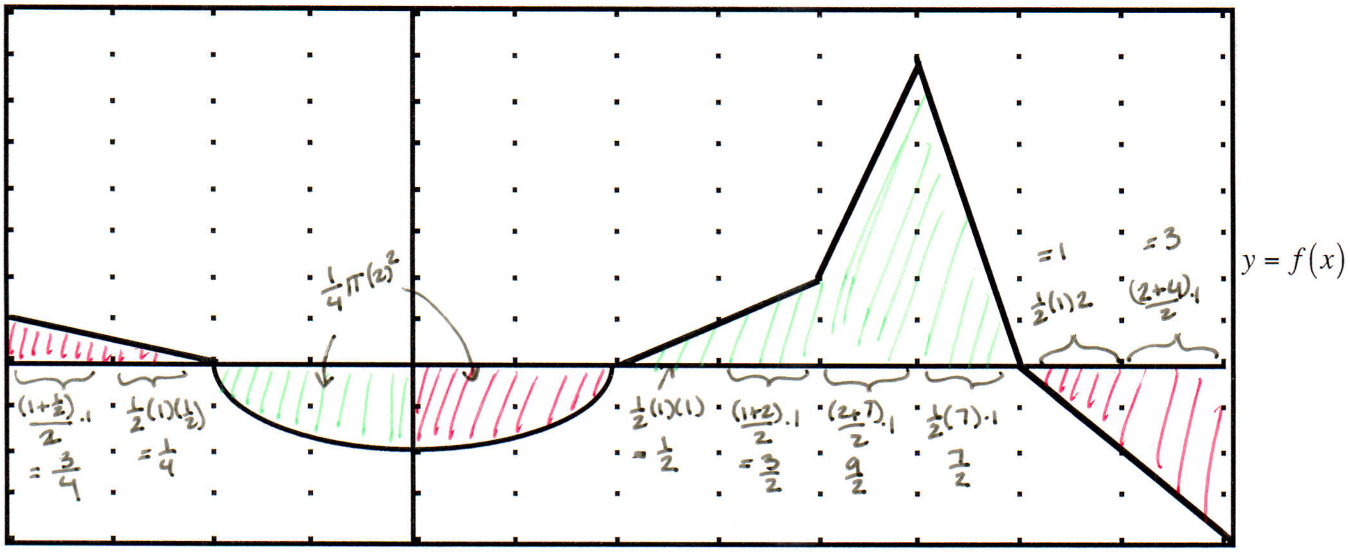
slope of f $= F'' = f'$ 0 − + −

0 1 3 4 8

f. Sketch a rough graph of $F(x)$

Use table of values from part a





2. Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(x)$ is above (the graph consists of lines and a semi-circle)

a. Complete the chart

area →

$F'(x)=f(x)$ y-values

x	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
$F(x)$	$\pi-1$	$\pi-\frac{1}{4}$	π		0		$-\pi$	$\frac{1}{2}-\pi$	$2-\pi$	$\frac{13}{2}-\pi$	$10-\pi$	$9-\pi$	$6-\pi$
$F'(x)$	1	1/2	0		-2		0	1	2	7	0	-2	-4

b. On what subintervals of $[-4,8]$ is F increasing? Decreasing?

F Inc dec Inc dec

F' (+ | − | + | −) with −4, −2, 2, 6, 8

Inc $(-4,-2)$ $(2,6)$ $F'>0$

Dec $(-2,2)$ $(6,8)$ $F'<0$

c. Where in the interval $[-4,8]$ does F achieve its minimum value? What is the minimum value? Justify answer.

min value occurs $\underbrace{\text{dec} \to \text{inc}}_{x=2}$ or boundary $x=0,8$ $\boxed{F(2)=-\pi}<0$ ← most negative value

$F(8)=6-\pi>0$

$F(0)=\pi-1>0$

d. Where in the interval $[-4,8]$ does F achieve its maximum value? What is the maximum value? Justify answer.

max value occurs inc → dec or boundary

$x=-2$

$x=6$

$F(-2)=\pi$

$\boxed{F(6)=10-\pi}$ ← most positive value

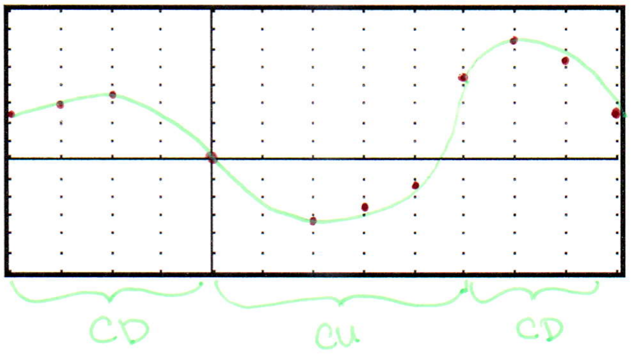
e. On what subintervals of $[-4,8]$ is F concave up and concave down? Find its inflection points. Justify answers.

F CD CU CD

Slope of f → f' = F'' (− | + | −) with −4, 0, 5, 8

f. Sketch a rough graph of $F(x)$

Use table of values from part a



H' °C/min

rate of temp

Question 2

← Calculator

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

$H'<0$

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

$\xleftarrow{}$ time

$$\frac{\text{area } H(t) \rightarrow}{10 \text{ min} \rightarrow} \quad \frac{°C \cdot \cancel{min}}{\cancel{min}}$$

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

$B(0)=100$

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

$B(10)=?$

a) $H'(3.5) \approx \frac{52-60}{5-2} = \frac{-8}{3} \approx -2.667$ °C/min

b) $\frac{1}{10}\int_0^{10} H(t)\,dt = \frac{1}{10}\left(\frac{66+60}{2}\cdot 2 + \frac{60+52}{2}\cdot 3 + \frac{52+44}{2}\cdot 4 + \frac{44+43}{2}\cdot 1\right)$

$\underbrace{\qquad\qquad}$ $= \frac{1}{10}\left(126 + 168 + 192 + \frac{87}{2}\right) = \frac{1059}{20} \approx 52.95°C$

Ave value of $H(t)$

$\frac{1}{10}\int_0^{10} H(t)\,dt = 52.95°C$ is the average temperature from $t=0$ to $t=10$ mins

c) $\int_0^{10} H'(t)\,dt = H(t)\Big]_0^{10} = H(10) - H(0) = 43 - 66 = -23°C$

$\int_0^{10} H'(t)\,dt$ is the drop in temp for the first 10 mins of cooling

d) $\underbrace{\int_0^{10} B'(t)\,dt}_{\text{Calc}} = B(10) - \underbrace{B(0)}_{100}$ $\Rightarrow B(10) = 34.18275$

$-65.81724 = B(10) - 100$

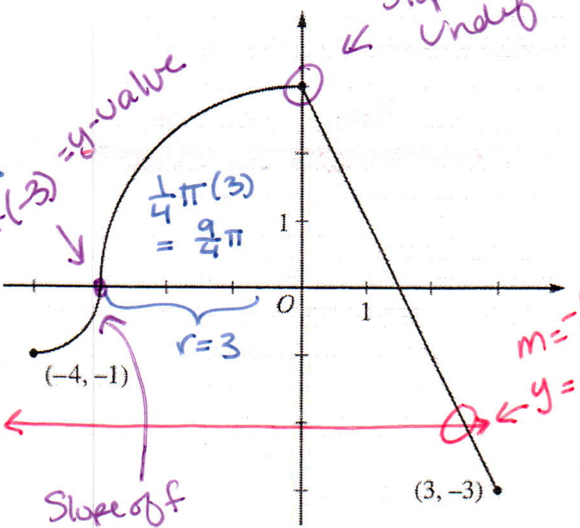
$H(10) - B(10) = 43 - 34.18275 = 8.81725$

How much cooler are the biscuits?

The Biscuits are 8.817°C cooler than the tea.

Question 4

← no Calc



Graph of f

The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above.

Let $g(x) = 2x + \int_0^x f(t)\,dt$. area under f

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer. inc → dec or boundary pt

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer. g'' changes sign

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

2 points slope start & stop

← ftc part 1

a)

$g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt$

$= -6 - \int_{-3}^{0} f(t)\,dt$

$= -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3)$

$= 2 + 0 = 2$

b)

$g'(x) = 2 + f(x)$

$0 = 2 + f(x)$

$f(x) = -2$ ← when does the graph of f have y-value = 2

$y = -2$ on $y = -2x + 3$

$-2 = -2x + 3$

$x = 5/2$

g: inc, dec ← max

g': + (−4 to 5/2), − (5/2 to 3)

−4, 5/2, 3

$g'(-3) > 0$ by part a

$g'(2.75) = 2 + \underbrace{f(2.75)}_{< -2} < 0$

max at x = 2.5 since the sign of g' changes from positive to negative

c)

g: cu, cu, cd

g'': +, +, −

−4, −3, 0, 3

$g'(x) = 2 + f(x)$

$g''(x) = f'(x) = 0$ Slope of f = 0 or undif

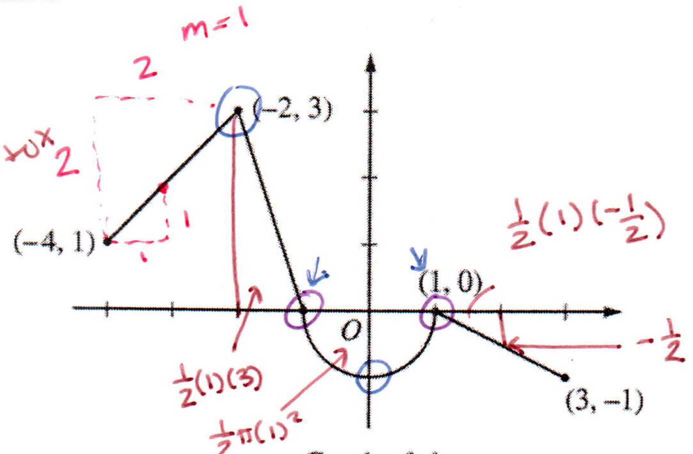
g has a point of inflection at x=0 since g''(x) changes sign at x=0.

d)

Ave rate of change $= \frac{f(3) - f(-4)}{3+4} = \frac{-3+1}{7} = -2/7$

To use the MVT the function must be differentiable, f' does not exist everywhere on [-4, 3] thus MVT does not apply

Question 3 ← no Calc



Graph of f

Let f be the continuous function defined on $[-4, 3]$ whose graph, consisting of three line segments and a semicircle centered at the origin, is given above. Let g be the function given by $g(x) = \int_1^x f(t)\,dt$. ✓ area from 1 to x

(a) Find the values of $g(2)$ and $g(-2)$. ✓

(b) For each of $g'(-3)$ and $g''(-3)$, find the value or state that it does not exist. ✓

(c) Find the x-coordinate of each point at which the graph of g has a horizontal tangent line. For each of these points, determine whether g has a relative minimum, relative maximum, or neither a minimum nor a maximum at the point. Justify your answers. ✓ ✓

g'=0
slope=0

(d) For $-4 < x < 3$, find all values of x for which the graph of g has a point of inflection. Explain your reasoning. ✓

g'' changes sign

a) $g(2) = \int_1^2 f(t)dt$

$= \frac{1}{2}(1)(-\frac{1}{2}) = -1/4$

$g(-2) = \int_1^{-2} f(t)dt = -\int_{-2}^1 f(t)dt$

$= -\left(\frac{3}{2} + \frac{\pi}{2}\right) = -\frac{(3+\pi)}{2} = \frac{-3-\pi}{2}$

b) $g'(x) = f(x)$ ← y-value of f

$g'(-3) = f(-3)$

$= 2$

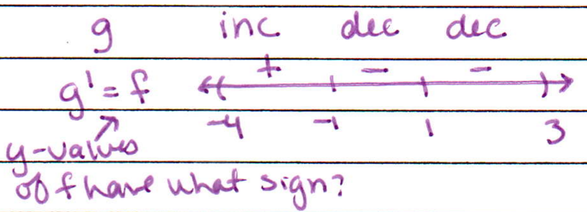
$g''(x) = f'(x)$ ← slope of f

$g''(-3) = f'(-3)$

$= 1$

c) $g'(x) = f(x) = 0$

$x = +1, -1$

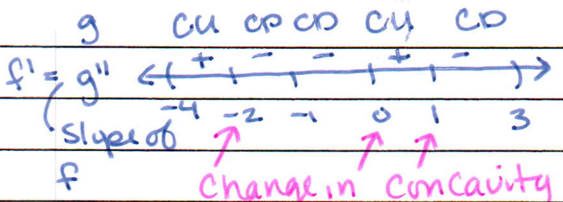


x = −1 is a relative maximum since g' changes from positive to negative. x = 1 is neither because g' does not change sign.

d) $g'' = f'(x) = 0$ or undef.

↗ slope of f is 0 or undf

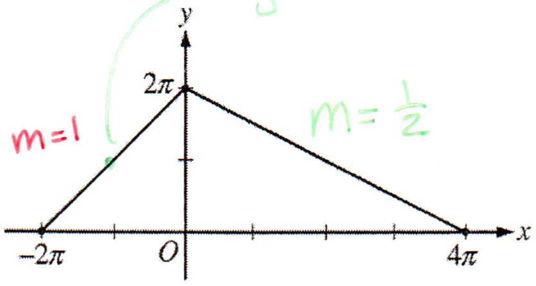
$x = -2, -1, 0, 1$



g has points of inflection at x = −2, 0, 1 because g'' changes sign at those values.

← no calc

Question 6



Graph of g

Let g be the piecewise-linear function defined on $[-2\pi, 4\pi]$ whose graph is given above, and let $f(x) = g(x) - \cos\left(\frac{x}{2}\right)$.

(a) Find $\int_{-2\pi}^{4\pi} f(x)\,dx$. Show the computations that lead to your answer.

(b) Find all x-values in the open interval $(-2\pi, 4\pi)$ for which f has a critical point. $f' = 0$ or undef

(c) Let $h(x) = \int_0^{3x} g(t)\,dt$. Find $h'\left(-\frac{\pi}{3}\right)$.

a)

$$\int_{-2\pi}^{4\pi} f(x)\,dx = \int_{-2\pi}^{4\pi}\left(g(x) - \cos\left(\tfrac{x}{2}\right)\right)dx = \overbrace{\int_{-2\pi}^{4\pi} g(x)\,dx}^{\text{area}} - \overbrace{\int_{-2\pi}^{4\pi} \cos\left(\tfrac{x}{2}\right)dx}^{\text{computation}}$$

$$\tfrac{1}{2}(6\pi)(2\pi) \quad - \quad 0$$

$$\boxed{6\pi^2}$$

$$\int_{-2\pi}^{4\pi} \cos\left(\tfrac{x}{2}\right)dx = 2\sin\left(\tfrac{x}{2}\right)\Big]_{-2\pi}^{4\pi} = 2\sin\left(\tfrac{4\pi}{2}\right) - 2\sin\left(\tfrac{-2\pi}{2}\right)$$

$$= 2\sin(2\pi) - 2\sin(-\pi)$$

$$= 2(0) - 2(0)$$

$$= 0$$

$\frac{d}{dx}\left[\sin\left(\frac{x}{2}\right)\right] = \cos\frac{x}{2}\cdot\frac{1}{2}$ ← balance

$$\boxed{\int_{-2\pi}^{4\pi} f(x)\,dx = 6\pi^2}$$

b)

$$f'(x) = \underset{\substack{\uparrow \\ \text{slope of } g \\ \text{changes at } x=0}}{g'(x)} + \tfrac{1}{2}\sin\left(\tfrac{x}{2}\right) = \begin{cases} 0 = 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right) & -2\pi < x < 0 \\ & \leftarrow f' \text{ DNE} \\ 0 = \frac{1}{2} + \frac{1}{2}\sin\left(\frac{x}{2}\right) & 0 < x < 4\pi \end{cases}$$

$0 = 1 + \frac{1}{2}\sin\left(\frac{x}{2}\right)$

$\sin\left(\frac{x}{2}\right) = -2$

never happens

$0 = \frac{1}{2} + \frac{1}{2}\sin\frac{x}{2}$

$\sin\left(\frac{x}{2}\right) = 1$ ← $\theta = \pi/2$ $\sin\theta = 1$

$\frac{x}{2} = \pi/2$

$x = \pi$

f has critical points at $x = 0$ + $x = \pi$.

c.

$$h(x) = \int_0^{3x} g(t)\,dt = G(t)\Big]_0^{3x} = G(3x) - G(0)$$

$$h'(x) = \longrightarrow g(3x)\cdot 3 - 0$$

$$h'(-\pi/3) = g\left(3\left(-\tfrac{\pi}{3}\right)\right)\cdot 3$$

$$= g(-\pi)\cdot 3$$

y-value on g → $= \pi\cdot 3$

$$= 3\pi$$