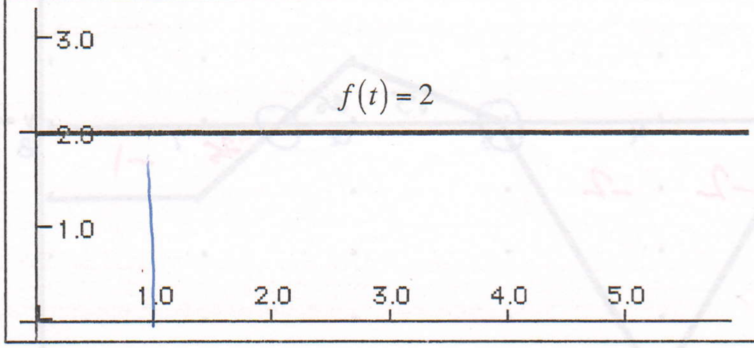
per 3

The Accumulation Function - Classwork

Now that we understand that the concept of a definite integral is nothing more than an area, let us consider a very special type of area problem. Suppose we are given a function $f(x)=2$. We know that to be a horizontal line at $y=2$. First, realize that the equation of the graph of $f(x)=2$ is the same as $f(t)=2$ is the same as $f(k)=2$. Whether we use *x*, *t*, or *k*, it does not matter. The graph is still a horizontal line. We are used to using *x* but we will see a good reason that we will occasionally be using another letter.

$\int 2\,dt$



Consider the expression $\int_0^x f(t)\,dt$

Above, we have the graph $f(t)=2$. We now want to consider the expression $\int_0^x f(t)\,dt$. What is this? It is a function of *x*. As *x* changes, $\int_0^x f(t)\,dt$ changes as well. Complete the chart below.

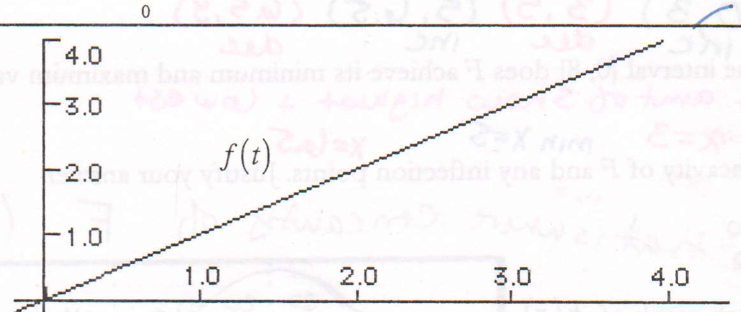
x	0	1	2	3	4	5	x
$\int_0^x f(t)\,dt$	$2(0-0)$ 0	$2(1-0)$ 2	$2(2-0)$ 4	$2(3-0)$ 6	$2(4-0)$ 8	$2(5-0)$ 10	$2(x-0)$ $2x$

It should be apparent that as *x* gets bigger, $\int_0^x f(t)\,dt$ increases as well. What we appear to be doing is "accumulating area" and we call $\int_0^x f(t)\,dt$ <u>the accumulation function</u>. Finally, it should be obvious why we use $f(t)=2$ to describe our line rather than $f(x)=2$. $\int_0^x f(x)\,dx$ would be confusing.

Let' s calculate the value of $\int_0^x f(t)\,dt$ for $x=0$ to 4

Complete the chart below

$y=x$



$\int x\,dx$

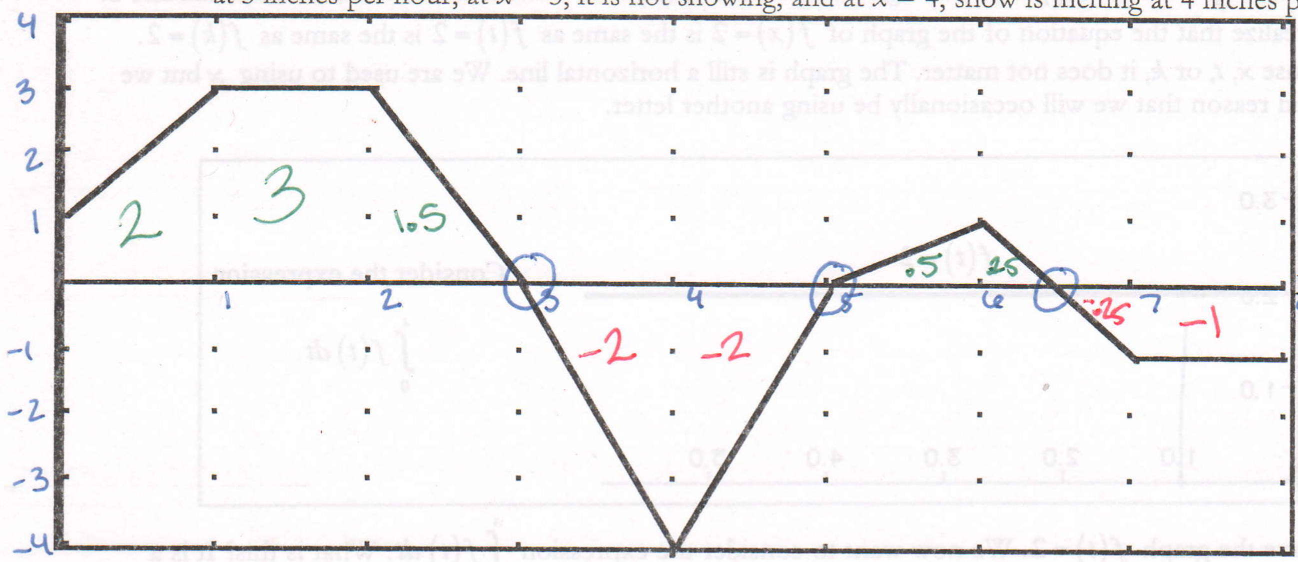
x	0	1	2	3	4	x
$\int_0^x f(t)\,dt$	$\frac{1}{2}(0)(0)$ 0	$\frac{1}{2}(1)(1)$ 1/2	$\frac{1}{2}(2)(2)$ 2	$\frac{1}{2}(3)(3)$ 9/2	$\frac{1}{2}(4)(4)$ 8	$\frac{1}{2}(x)(x) = \frac{x^2}{2}$

$f(x)\ \frac{in}{hr}$ x hrs $\int_1^2 f(x)\,dx = \frac{3\ in}{hr} \cdot 1\ hr = 3\ in$

from $t=1$ to 2 3 in of snow has fallen

Example 1) Let $F(x) = \int_0^x f(t)\,dt$ where the graph of $f(x)$ is below. Remember $f(x)$ is the same thing as $f(t)$.

Think of $f(x)$ as the rate of snowfall over a period of time. For instance, at $x = 1$, snow is falling at 3 inches per hour, at $x = 3$, it is not snowing, and at $x = 4$, snow is melting at 4 inches per hour.



$y = f(x) = F'(x)$

$f'(x) = F''(x)$

a. Complete the chart. In the snow analogy, $F(x)$ represents the accumulation of snow over time.

area under f

x	0	1	2	3	4	5	6	6.5	7	8
$F(x)$	0	2	2+3 5	6.5	4.5	2.5	3	3.25	3	2

amt of snow

b. Now let's consider $F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$. If we take the derivative of an integral, what would you expect to happen? Undo So $F'(x) = \frac{d}{dx}\int_0^x f(t)\,dt$ is the same thing as $f(x)$.

Knowing that, let's complete the chart.

y-value of f

x	0	1	2	3	4	5	6	6.5	7	8
$F'(x)$	1	3	3	0	-4	0	1	0	-1	-1

c. On what subintervals of [0, 8] is F increasing? Decreasing?

amt of snow

(0, 3) inc (3, 5) dec (5, 6.5) inc (6.5, 8) dec

d. Where in the interval [0, 8] does F achieve its minimum and maximum value? What are those values?

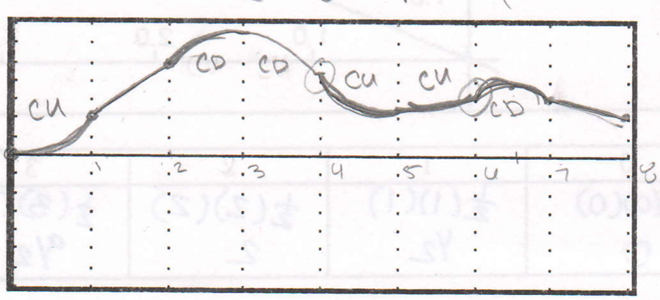
→ where is amt of snow highest + lowest

max $x=3$ min $x=5$ $x=6.5$ min $x=0, 8$

f. Find the concavity of F and any inflection points. Justify your answer

$x=4, 6$ because that is where concavity of F (look at F'')

h. Sketch a rough graph of $F(x)$



F CU no curvature CD

F' inc dec f

F'' + 0 − f'

f F'

pos slope = CU

neg slope = CD

0 slope = no concavity

F' F''

→ ↑ positive

← ↓ negative

Example 2) Let $F(x) = \int_0^x f(t)\,dt$ where f is the function graphed below (consisting of lines and a semi-circle)

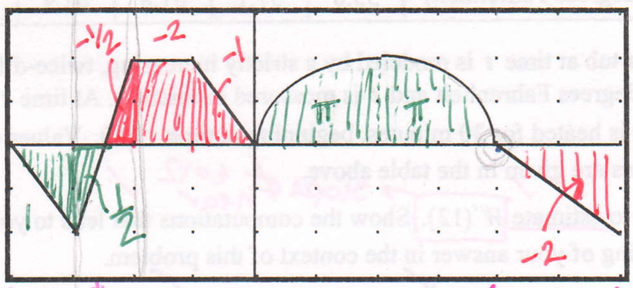
$r = 2$

$\frac{\pi(2)^2}{2} = 2\pi$

area

$F'(x) = f(x)$

y-values



Find the following:

F Dec Inc Dec

F' − +

a) $F(0)$ = 0

b) $F(2)$ = π ≈ 3.14

c) $F(4)$ = 2π ≈ 6.28

d) $F(6)$ = 2π − 2 ≈ 4.28

e) $F(-1)$ = −1

f) $F(-2)$ = −3

g) $F(-3)$ = −3

h) $F(-4)$ = −2

F(−2.5) = −3.5

i) $F'(4)$ = f(4) = 0

j) $F'(2)$ = 2

k) $F'(6)$ = −2

l) $F'(-3)$ = −2

F'(−1) = f(−1) = 2

m) On what subintervals of $[-4,6]$ is F increasing and decreasing. Justify your answer.

Dec (−4, −2.5) inc (−2.5, 4) Dec (4, 6)

F dec inc inc dec

F' − + + − (−2.5, 0, 4)

slope of F

n) Where in the interval $[-4,6]$ does F achieve its minimum value? What is the minimum value?

min Value is −3.5, at x = −2.5

where area is most negative

o) Where in the interval $[-4,6]$ does F achieve its maximum value? What is the minimum value?

max value is 2π, at x = 4

−3

p) Where on the interval $[-4,6]$ is F concave up? Concave down? Justify your answer.

→ F''

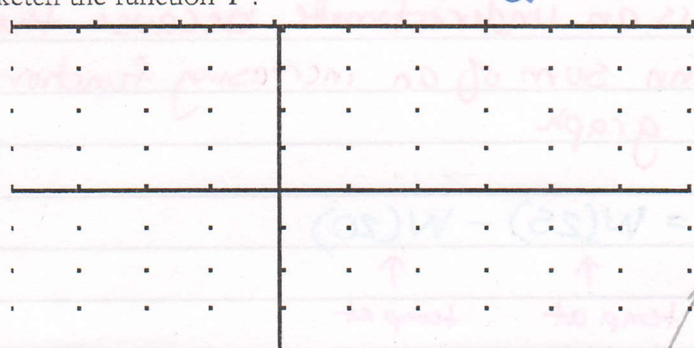
q) Where does F have points of inflection?

→ change in concavity

F'' = f'(x)

slope of f = F'

r) Sketch the function F.



each tic mark on y-axis is 2 units

F: CD CU concavity straight line CD CU CD

F'': − + 0 − + −

−4 −3 −2 −1 0 2 6

slope of f

$W'(t) =$ slope of W

°F/min — rate of temp

Question 1

$W'(12)$ = slope

t (minutes)	0	4	9	15	20	← base
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0	height

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

$W(0) = 55$

(a) Use the data in the table to estimate $\boxed{W'(12)}$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

→ slope near $t=12$

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

rectangle

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$? ← $W(25)$

a) $W'(12) \approx \frac{67.9 - 61.8}{15 - 9} = \frac{6.1}{6} = 1.016$ °F/min

at $t=12$ the temp is rising at a rate of 1.016 °F/min

b) $\int_0^{20} W'(t)\,dt = W(t)\Big]_0^{20} = W(20) - W(0) = 71 - 55 = 16$°F

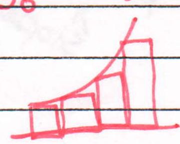
FTC!

$\int_0^{20} W'(t)\,dt$ is the rise in temp, 16°F, during the first 20 mins.

→ is the temp accumulated from $t=0$ to $t=20$.

c) $\frac{1}{20}\int_0^{20} W(t)\,dt = \frac{1}{20}(4\cdot 55 + 5\cdot 57.1 + 6\cdot 61.8 + 5\cdot 67.9) = 60.79$°F

$\frac{1}{\text{min}}$ min·°F



$\frac{1}{20}\int_0^{20} w(t)\,dt$ is an underestimate because the left Riemann sum of an increasing function is below the graph

d) $\underbrace{\int_{20}^{25} W'(t)\,dt}_{\text{temp accumulated from 20 to 25 min}} \overset{\text{FTC}}{=} W(t)\Big]_{20}^{25} = W(25) - W(20)$

temp at $t=25$ — temp at $t=20$

$2.04315 = W(25) - 71$

$W(25) = 73.04315$ °F

→ 73.043