AP® CALCULUS AB
2011 SCORING GUIDELINES

$W(0)=1400$ **Question 5** $W'>0$

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010. $y-y_1=\frac{dw}{dt}(t-t_1)$ $(0,1400)$

(a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).

(b) Find $\frac{d^2W}{dt^2}$ in terms of W. Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.

(c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$. → Separate, integrate, +C, initial cond, solve for W

a) $(0,1400)$ $\left.\frac{dW}{dt}\right|_{W=1400} = \frac{1}{25}(1400-300) = \frac{1100}{25} = 44$

$y-1400 = 44(t-0) \rightarrow \boxed{y = 44t + 1400}$

$W(1/4) \approx y(1/4) = 44(1/4) + 1400 = 11 + 1400 = \boxed{1411 \text{ tons}}$

b) $\frac{d}{dt}\left(\frac{dw}{dt}\right) = \frac{d^2w}{dt^2} = \frac{1}{25}\left(\frac{dw}{dt}\right) = \frac{1}{25}\left(\frac{1}{25}(W-300)\right) = \boxed{\underbrace{\frac{1}{625}}_{>0}\underbrace{(W-300)}_{\text{smallest } W \text{ is } 1400 \text{ + W increases only } >0}}$

Since $\frac{d^2w}{dt^2} > 0$ the graph of W is concave up making part (a) an underestimate

c) $\int \frac{1}{W-300}\, dW = \int \frac{1}{25}\, dt$

$\ln|W-300| = \frac{1}{25}t + C$ always >0

$e^{\frac{t}{25}+C} = W - 300$

$Ce^{t/25} + 300 = W$

$Ce^{0/25} + 300 = 1400$

$C + 300 = 1400$

$C = 1100$

$W = 1100e^{t/25} + 300$

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2010 SCORING GUIDELINES

Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

$m =$

(a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.

$y > 0$

(b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.

(c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

separate, integrate, +C, solve for y, initial cond.

a) $(1,2)$ $\quad \left.\frac{dy}{dx}\right|_{(1,2)} = (1)2^3 = 8$ $\quad \boxed{y-2 = 8(x-1)}$

b) $f(1.1) \approx y(1.1) = 8(1.1-1) + 2$ $\quad \boxed{f(1.1) \approx 2.8}$

$= 8(.1) + 2 = 2.8$

$$\frac{d^2y}{dx^2} = \underbrace{y^3}_{y>0}\underbrace{(1 + 3x^2y^2)}_{>0} > 0$$

($x^2 > 0$, $y^2 > 0$)

$y = f(x)$ is concave up so the approximation is less than $f(1.1)$

c) $\int \frac{1}{y^3}\,dy = \int x\,dx$ $\qquad \frac{1}{y^3} = y^{-3}$

$-2\left(\frac{y^{-2}}{-2} = \frac{x^2}{2} + C\right)$

$\frac{1}{y^2} = -x^2 + C$ $\qquad (1,2)$

$\frac{1}{4} = -1^2 + C$

$C = \frac{4}{4} + \frac{1}{4} = 5/4$

$\frac{1}{y^2} = -x^2 + 5/4$

$\frac{4 \cdot 1}{4 \cdot (\frac{5}{4} - x^2)} = y^2$

$\frac{4}{5-4x^2} = y^2$

$y = \pm\frac{\sqrt{4}}{\sqrt{5-4x^2}}$

$y = \pm\frac{2}{\sqrt{5-4x^2}}$

$\boxed{y = +\frac{2}{\sqrt{5-4x^2}}}$

Since $(1,2)$ $+$

AP® CALCULUS AB
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Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

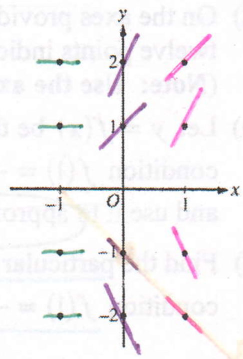
(Note: Use the axes provided in the exam booklet.)

(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

$x+1=0$
$x=-1$ } zero slopes

a)



$x=0$
$\to \frac{1}{y}$

$x=1$
$\to \frac{2}{y}$

a) See diagram

b) $\frac{dy}{dx} = \frac{x+1}{y} = -1 \quad y \neq 0$

$x+1 = -y$

$\boxed{y=-x-1}$ is the set of all (x,y) such that $\frac{dy}{dx} = -1$

c) $\int y\,dy = \int (x+1)\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$y^2 = x^2 + 2x + C$

$(0,-2)$

$(-2)^2 = 0^2 + 2(0) + C$

$C = 4$

$y^2 = x^2 + 2x + 4$

$y = \pm\sqrt{x^2+2x+4}$

$\boxed{y = -\sqrt{x^2+2x+4}}$

since $(0, -2)$

$-$

Question 6

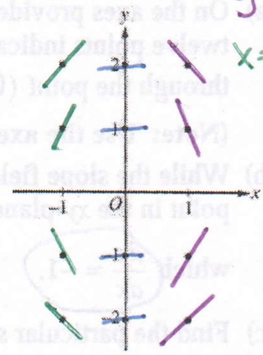
Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

a) $x=0$ $\frac{dy}{dx}=0$

$x=1 \rightarrow -\frac{2}{y}$

$x=-1 \rightarrow \frac{2}{y}$

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
(Note: Use the axes provided in the pink test booklet.)



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1, -1)$ and use it to approximate $f(1.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

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a) see diagram

b) $(1,-1)$ $\left.\frac{dy}{dx}\right|_{(1,-1)} = \frac{-2(1)}{-1} = 2$ $\boxed{y+1 = 2(x-1)}$

$f(1.1) \approx y(1.1) = 2(1.1-1)-1 = 2(.1)-1 = -.8$ $\boxed{f(1.1) \approx -.8}$

c) $\int y\,dy = \int -2x\,dx$

$\frac{y^2}{2} = -x^2 + C$

$y^2 = -2x^2 + C$ $(1,-1)$

$(-1)^2 = -2(1) + C$

$C = 3$

$y^2 = -2x^2 + 3$

$y = \pm\sqrt{3-2x^2}$

$\boxed{y = -\sqrt{3-2x^2}}$

since $(1,-1)$ negative