Calculus
18 Optimization

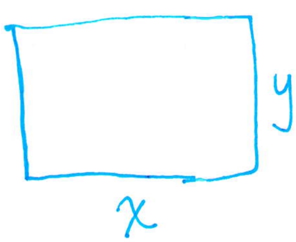
Name: per 3
Date: ____________ Per: _____

The most common applications of calculus involves finding max/min. This section is about applying those concepts to real world problems.

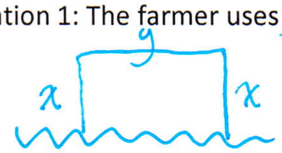
Steps:

1) DRAW A PICTURE (if possible). Assign symbols to all given quantities and quantities to be determined.
2) Write a **primary equation** for the quantity to be optimized (maximized or minimized). what is being optimized
3) Reduce the primary equation to a single variable, often this requires the use of a **secondary equation** that relates the variable in the primary equation. → helper eqn → constraint
4) Determine the domain (feasible input values) for the primary equation. E.g. you can't have a sheet of paper that is -6 inches.
5) Find the max/min using calculus. i.e. first derivative test. On a closed interval a max/min always exist.

Example 1: A farmer has 500 ft. of fencing (→ perimeter) with which to build a rectangular corral. What is the maximum area he can enclose with the fencing? CLASSIC

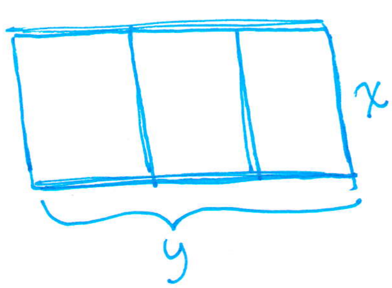
<table>
<tr>
<td rowspan="3">Diagram:

Feasible Domain: $0 < x < 250$</td>
<td>Primary Equation:
$A = x \cdot y$</td>
<td rowspan="3">Find the max/min:
$A' = 250 - 2x$
$0 = 250 - 2x$
$x = 125$
A'   +   −
CP ← 125 →
A inc dec
max @ $x = 125$
$y = 250 - 125 = 125$
max Area $= (125)(125)$
$= \boxed{15{,}625 \text{ ft}^2}$</td>
</tr>
<tr>
<td>Secondary Equation:
$500 = 2x + 2y$
$250 = x + y$
$\rightarrow y = 250 - x$</td>
</tr>
<tr>
<td>Function to be optimized:
$A = x(250 - x) = 250x - x^2$
$A' = 250 - 2x$</td>
</tr>
</table>

Variation 1: The farmer uses a river as one side of the corral



$500 = 2x + y$

Variation 2: The farmer wants to place two internal dividers in the corral so that there are three sections.

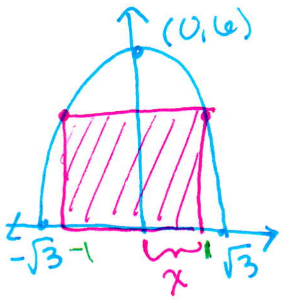


$500 = 2y + 4x$

Example 2: What is the maximum area of a rectangle with a base that lies on the $x-axis$ and the two upper vertices lie on the graph of $y = 6 - 2x^2$. CLASSIC

$6-2x^2=0$
$6=2x^3$
$x=\pm\sqrt{3}$

Diagram:



Feasible Domain:

$0 < x < \sqrt{3}$

Primary Equation:

$A = \underbrace{2x}_{w} \cdot y$

Secondary Equation:

$y = 6 - 2x^2$

Function to be optimized:

$A = 2x\,(6-2x^2)$

$A = 12x - 4x^3$

Find the max/min:

$A' = 12 - 12x^2$

$0 = 12 - 12x^2$

$x = \pm 1$

A'		+		−	
CP	0		1		$\sqrt{3}$
A		inc		dec	

↳ max @ $x=1$

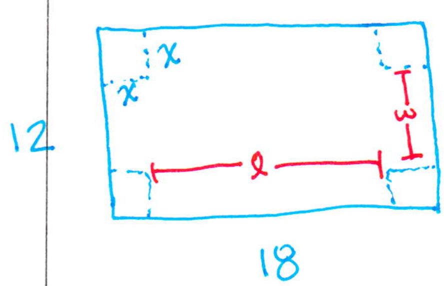
$y = 6 - 2(1)^2 = 4$

max Area $= 2(1)(4)$

$= \boxed{8}$

Example 3: An open top box is to be made from a rectangular piece of sheet metal that measures 12 in by 18 in. What is the maximum volume of such a box? CLASSIC

Diagram:



Feasible Domain:

$0 < x < 6$

Primary Equation:

$V = \ell \cdot w \cdot x$

Secondary Equation:

$\ell = 18 - 2x$

$w = 12 - 2x$

Function to be optimized:

$V = (18-2x)(12-2x)\cdot x$

$V = 4x^3 - 60x^2 + 216x$

Find the max/min:

$V' = 12x^2 - 120x + 216$

$0 = 12(x^2 - 10x + 18)$

$x = \dfrac{+10 \pm \sqrt{(-10)^2 - 4(1)(18)}}{2(1)}$

$x = 5 \pm \sqrt{7}$

$x = \underbrace{5-\sqrt{7}}_{\text{max}},\ 5+\sqrt{7}$

$x \approx 2.35$

$\ell = 18 - 2(2.35)$

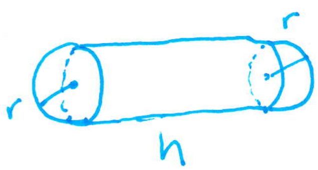
$w = 12 - 2(2.35)$

max Volume $= (2.35)(\ \ )(\ \ )$

$\boxed{= 228.162}$

Example 4: A industrial tank is in the shape of a right cylinder with two hemispheres on either end. The tank must have a volume of 3000 cubic ft. The hemispherical ends cost twice as much per square foot of surface area as the sides. Find the dimensions that will minimize the cost.

Diagram:



$\frac{3}{4\pi} \cdot 3000 = \frac{4}{3}\pi r^3 \cdot \frac{3}{4\pi}$

$\sqrt[3]{\frac{9000}{4\pi}} = \sqrt[3]{r^3}$

Feasible Domain:

$0 < r < \sqrt{\frac{9000}{4\pi}}$

8.9

Primary Equation:

$S = 2\pi r h + (4\pi r^2)2$

Secondary Equation:

$3000 = \pi r^2 h + \frac{4}{3}\pi r^3$

$h = \frac{3000 - \frac{4}{3}\pi r^3}{\pi r^2}$

Function to be optimized:

$S = 2\pi r\left(\frac{3000 - \frac{4}{3}\pi r^3}{\pi r^2}\right)$

$+ 8\pi r^2$

$S = \frac{6000 - \frac{8}{3}\pi r^3}{r} + 8\pi r^2$

↓ common denom.

$S = \frac{6000}{r} - \frac{8}{3}\pi r^2 + \frac{24}{3}\pi r^2$

$S = \frac{6000}{r} + \frac{16}{3}\pi r^2$

$= 6000 r^{-1}$

Find the max/min:

$S' = \frac{-6000}{r^2} + \frac{32}{3}\pi r$

$r \neq 0$

$\left(0 = \frac{-6000}{r^2} + \frac{32}{3}\pi r\right) 3r^2$

$0 = -18{,}000 + 32\pi r^3$

$32\pi r^3 = 18000$

$r = \sqrt[3]{\frac{18{,}000}{32\pi}} \approx 5.6$

S'		−	+
CP	0	5.6	8.9
S		dec	inc

min @ $r = 5.6$

$h = 22.5$

$\boxed{S = 1579.84}$

Example 5: Find two positive integers whose product is a maximum if the sum of those numbers is 56.
CLASSIC

Diagram:

N/A

Feasible Domain:

$0 < x < 56$

Primary Equation:

$P = x \cdot y$

Secondary Equation:

$56 = x + y$

$\rightarrow y = 56 - x$

Function to be optimized:

$P = x(56 - x)$

$P = 56x - x^2$

Find the max/min:

$P' = 56 - 2x$

$0 = 56 - 2x$

$x = 28$

P'		+	−
CP	0	28	56
P		inc	dec

max at 28

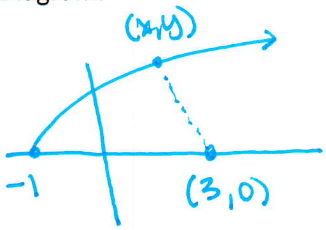
$y = 56 - 28 = 28$

max prod $= (28)(28)$

$\boxed{= 784}$

Example 6: Find the point on the graph of $y = \sqrt{x+1}$ closest to the point $(3, 0)$.

Diagram:



Feasible Domain:

any $x > 0$

Primary Equation:

$$d = \sqrt{(x-3)^2 + (y-0)^2}$$

$$d = \sqrt{x^2 - 6x + 9 + y^2}$$

Secondary Equation:

$$y = \sqrt{x+1}$$

Function to be optimized:

$$d = \sqrt{x^2 - 6x + 9 + (\sqrt{x+1})^2}$$

$$d = \sqrt{x^2 - 6x + 9 + x + 1}$$

$$d = \sqrt{x^2 - 5x + 10}$$

Find the max/min:

$$d' = \frac{1}{2}(x^2 - 5x + 10)^{-1/2}(2x - 5)$$

$$0 = \frac{2x-5}{2\sqrt{x^2-5x+10}}$$

no real roots

$$x = 5/2$$

d′: − +

CP: $5/2$

d: dec inc

min at $x = 5/2$

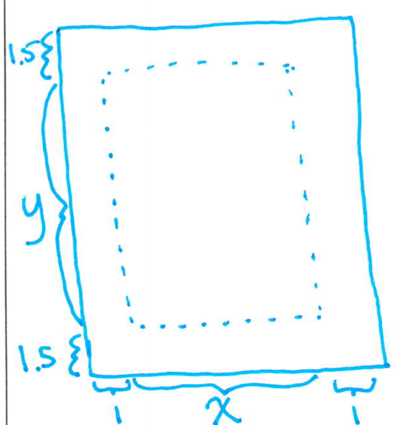
$$y = \sqrt{\frac{5}{2}+1} = \sqrt{\frac{5}{2}+\frac{2}{2}}$$

$$y = \sqrt{7/2}$$

Closest point $\left(\frac{5}{2}, \sqrt{\frac{7}{2}}\right)$

Example 7: A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be $1\frac{1}{2}$ inches, and the margins on the left and right are to be 1 in. What should the dimensions of the paper be so that the least amount of paper is used?

Diagram:



Feasible Domain:

$0 < x < 24$

Primary Equation:

$$A = (y+3)(x+2)$$

Secondary Equation:

$$24 = x \cdot y$$

$$y = \frac{24}{x}$$

Function to be optimized:

$$A = \left(\frac{24}{x} + 3\right)(x+2)$$

$$A = 3x + \frac{48}{x} + 30$$

Find the max/min:

$$A' = 3 - \frac{48}{x^2}$$

$$\left(0 = 3 - \frac{48}{x^2}\right)x^2 \quad x \neq 0$$

$$0 = 3x^2 - 48$$

$$x^2 = 16$$

$$x = \pm 4 \quad (-4 \text{ not in domain})$$

A′: − +

CP: 0, 4, 24

A: dec inc

min at $x = 4$

$$y = \frac{24}{4} = 6$$

Least Amt of paper $= (6+3) \times (4+2)$

$$\boxed{9 \times 6}$$