Intervals

(a, b) is the open interval from x=a to x=b, this interval does NOT include the points a and b.

[a, b] is the closed interval from x=a to x=b, this interval DOES include the points a and b.

(a, b] and [a, b) are the CLOPEN (aka half open or half closed) interval containing one, but not both of the endpoints.

yes Maceo, clopen
is a word

$\pm\infty$ can be considered both open and closed

Types of Discontinuities

revisited with calculus

Infinite Discontinuity (a.k.a. vertical asymptote, a.k.a non removable discontinuity) - a denominator that cannot be reduced.

the limit is infinite as x approaches c $\lim_{x \to c} f(x) = \pm\infty$

Point Discontinuity (a.k.a. hole, a.k.a. removable discontinuity) - a denominator that can be reduced.

the limit exists but does not equal f(c)

Jump Discontinuity - usually a piecewise function located at the boundary point.

the limit from the left and right do not agree

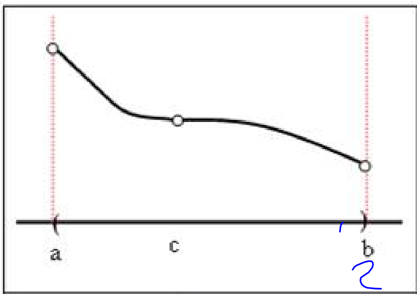


Fig. a
Function not defined at c

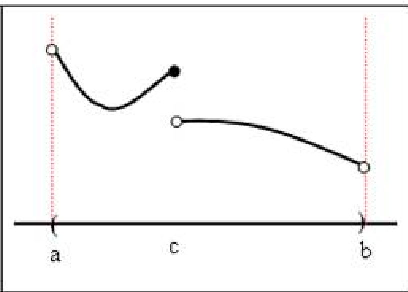


Fig. b
A Jump Discontinuity:
Function exists at c, but the limit doesn't exist at c

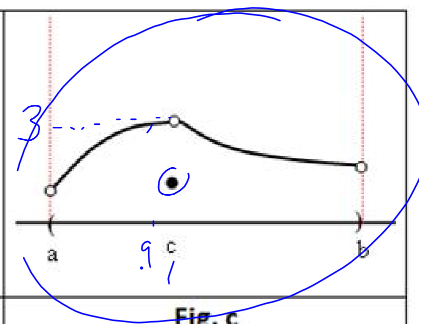


Fig. c
A Removable or Point Discontinuity
Both function and limit exist at c, but they are not equal to one another

Types of Continuity

Continuity at a point - a function f is continuous at a point, x=c, if the following THREE conditions are met:

1. $\lim_{x \to c} f(x) exists$ eliminating jump and oscillating

2. $f\left(c\right) is\, defined$ eliminating infinite

3. $\lim_{x \to c} f(x) = f(c)$ eliminates point

Continuity on an open interval - a function is continuous on an open interval (a, b) if the function is continuous at every point in the interval.

Left Continuous - $\lim_{x \to c^-} f(x) = f(c)$

Right Continuous - $\lim_{x \to c^+} f(x) = f(c)$

5 Limits and Continuity

Ex 1) Describe the type of discontinuities using limits.

$$f(x)=\frac{x^2-5x+6}{x^2-9}=\frac{\cancel{(x-3)}(x-2)}{(x+3)\cancel{(x-3)}}$$

$x-3=0$
$x=3$

$$\lim_{x\to 3}\frac{x-2}{x+3}=\frac{1}{6}$$

$$\lim_{x\to -3}\frac{x-2}{x+3}=\text{DNE}$$

$f(3)=\text{Undef}$

$x=3$ is a point dis.

$x=-3$ is an infinite disc.

Its called a REMOVABLE discontinuity because we can REMOVE the discontinuity. e.g.

$$g(x)=\begin{cases} f(x) & x\neq 3,-3 \\ \frac{1}{6} & x=3 \\ \cancel{7} & \cancel{x=-3} \end{cases}$$

plugging the hole

Ex 2) Describe the discontinuities using limits

$$f(x)=\begin{cases} x^2-1 & x<2 \\ 2x+4 & x\geq 2 \end{cases}$$

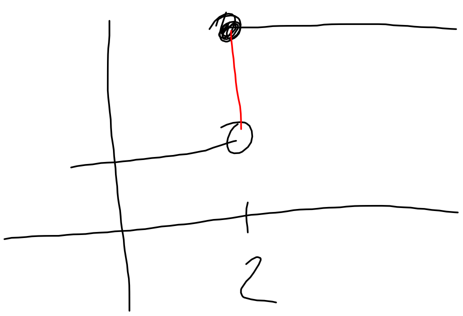
$$\lim_{x\to 2^-}(x^2-1) \qquad \lim_{x\to 2^+} 2x+4$$

$$= 2^2-1 \qquad = 2(2)+4$$

$$= 3 \qquad = 8$$

f(x) has a jump disc at $x=2$

b/c $\lim_{x\to 2^-} f(x)=3$ + $\lim_{x\to 2^+} f(x)=8$



$$f(x) = \begin{cases} x^2+ax+b & x < -1 \\ x^2+2b+1 & -1 \leq x \leq 2 \\ 3x-a & x > 2 \end{cases}$$

$$\lim_{x\to -1^-} x^2+ax+b = \lim_{x \to -1^+} x^2+2b+1$$

$$1-a+b = 1+2b+1$$

$$-a = +1+b$$

$$a = -1-b$$

$$\lim_{x\to 2^-} x^2+2b+1 = \lim_{x\to 2^+} 3x-a$$

$$-1-b = -2b+1$$

$$2b+5 = 6-a$$

$$a = -2b+1$$

$$b = 2$$

$$a = -3$$

$$b = \frac{a-1}{-2}$$

Ex 3) Find the value of ~~a~~ c that would make the function continuous.

1. $\lim_{x \to c} f(x) exists$ ✓
2. $f(c) is defined$ ✓
3. $\lim_{x \to c} f(x) = f(c)$

$$f(x) = \begin{cases} x^2 - \cancel{c}^{\,5/3} & x < 5 \\ 4x + 2\cancel{c}^{\,10/3} & x \geq 5 \end{cases}$$

$$\lim_{x \to 5^-} x^2 - c = \lim_{x \to 5^+} 4x + 2c$$

$$25 - c = 20 + 2c$$

$$5 = 3c$$

$$c = 5/3$$

value that makes the limits =

$$\lim_{x \to 5} f(x) = \frac{75}{3} - \frac{5}{3} = \boxed{\frac{70}{3}}$$ ★1

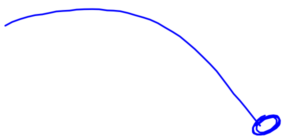
$$f(5) = \left\{ 4(5) + \frac{10}{3} \quad x \geq 5 \right.$$

$$f(5) = \frac{70}{3}$$ ★2

$$\lim_{x \to 5} f(x) = f(5)$$

$$\frac{70}{3} = \frac{70}{3}$$ ★3

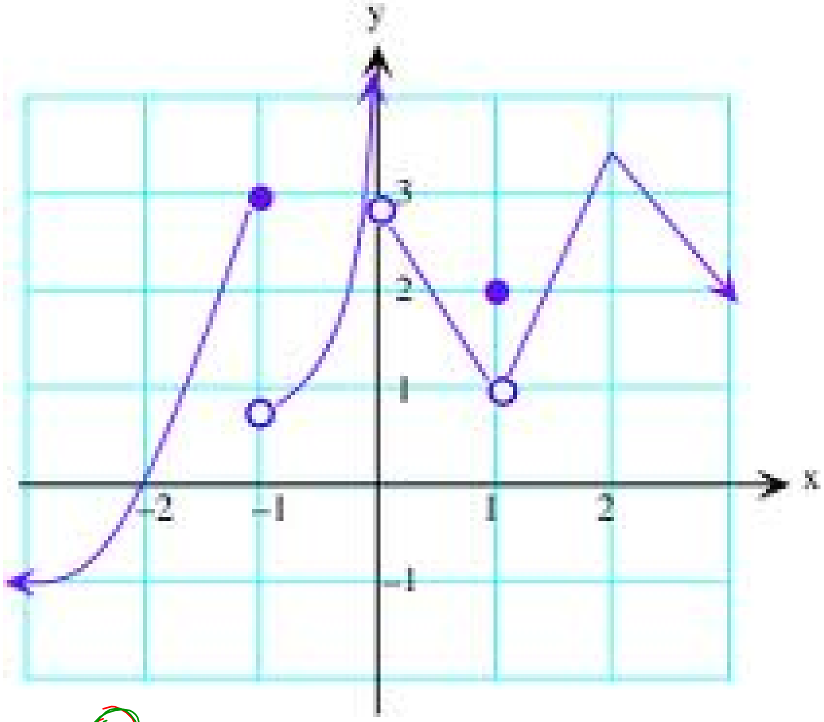
$$f(x) = \begin{cases} \cancel{\dfrac{(x-5)(x+2)}{(x-5)}} & x < 5 \\ & x > 5 \end{cases}$$



$$\frac{1}{x-5}$$

$$x < 3$$

Ex 4) Describe the continuity and discontinuity of the graph



$x = -1, 0, 1$

$\lim_{x \to -1^-} f(x) = 3$ $\lim_{x \to -1^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) =$ $\lim_{x \to 0^+} f(x) =$

Ex 5) Determine if the function is continuous at the boundary point.

$$f(x)=\begin{cases} 3-x & x<2 \\ 1 & x=2 \\ x^2-3 & x>2 \end{cases}$$

$$\lim_{x\to 2^-} 3-x \overset{?}{=} \lim_{x\to 2^+} x^2-3$$

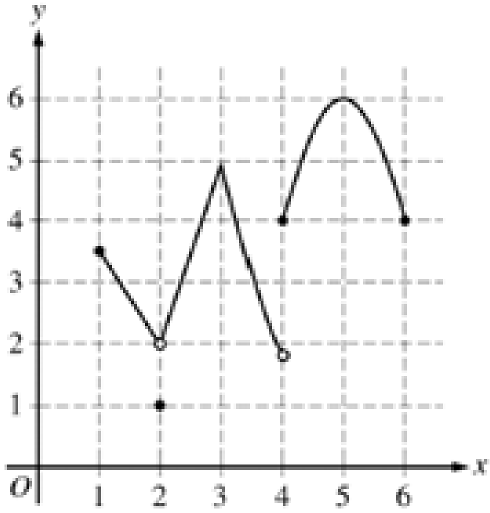
$$1 = 1$$

$$\lim_{x\to 2} f(x) = 1 \quad 1$$

$$f(2) = 1 \quad 2$$

$$\lim_{x\to 2} f(x) = f(2) \quad 3$$

$$1 = 1$$



Graph of f

5. The graph of the function f is shown above. Which of the following statements is false?

T (A) $\lim_{x \to 2} f(x)$ exists. $x \to 2$

T (B) $\lim_{x \to 3} f(x)$ exists. $x \to 3$

F (C) $\lim_{x \to 4} f(x)$ exists. $x \to 4$

T (D) $\lim_{x \to 5} f(x)$ exists. $x \to 5$

T (E) The function f is continuous at $x = 3$.

cts @ 3

$$f(x) = \begin{cases} \dfrac{(2x+1)(x-2)}{x-2} & \text{for } x \neq 2 \\ k & \text{for } x = 2 \end{cases}$$

Let f be the function defined above. For what value of k is f continuous at $x = 2$?

$$f(x) = \begin{cases} \dfrac{(2x+1)\cancel{(x-2)}}{\cancel{(x-2)}} & x \neq 2 \\ k & x = 2 \end{cases}$$

what type of disc. is present w/o k

point

$$\lim_{x \to 2} \frac{(2x+1)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$\lim_{x \to 2} 2x+1 = \qquad 2(2)+1 = 5$$

$$k = 5$$

INTERMEDIATE VALUE THEOREM.

x-values

y

If f is a function that is continuous over the interval $[a, b]$ and m is some number between $f(a)$ and $f(b)$, then there $\boxed{\text{exists}}$ a number c between a and b such that $f(c) = m$.

y-values

x

existance theorem

Example 1) Is any real number exactly 1 less than its cube?

$1 \stackrel{?}{=} 1^3 - 1$ ✗

$2 \stackrel{?}{=} 2^3 - 1$ ✗

$\pi \stackrel{?}{=} \pi^3 - 1$ ✗

$x \stackrel{?}{=} x^3 - 1 \rightarrow 0 = x^3 - x - 1$

$f(x) = x^3 - x - 1$

$1.32^3 - 1$

$\boxed{x = 1.32471}$

IVT: $f(x) = x^3 - x - 1$ is a polynomial, thus $f(x)$ is cts.

W.t.s. (want to show) $f(c) = 0$ so that $c = c^3 - 1$ is true

$f(1) = -1$

$f(2) = 5$

Since f is cts, and $f(1) < 0 < f(2)$, there exists a "c" in $[1, 2]$ such that $f(c) = 0$.

$x = x^3 - 1 \qquad\qquad 0 = x^3 - x - 1$

$1 = x^3 - x \qquad\qquad f(x) = x^3 - x - 1$

$g(x) = x^3 - x$

$g(1) = 0 \quad 1$

$g(2) = 6$

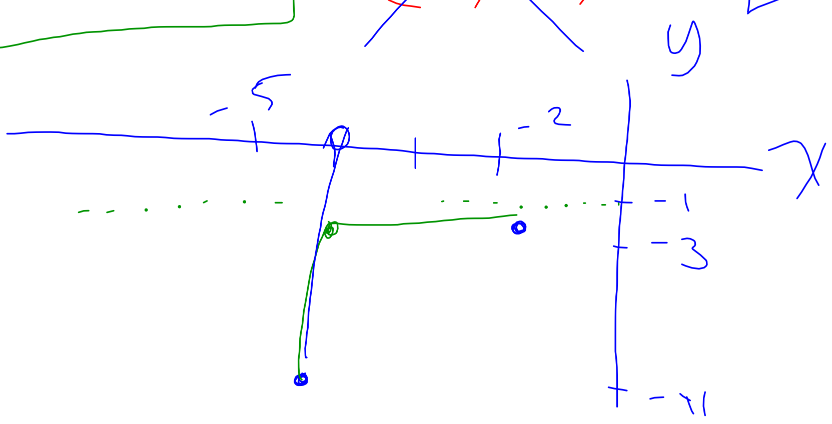
Example 2) Let f(x) be continuous on [-5, -2] with some of the values shown on the following table:

x	-5	-4	-2
f(x)	-11	a	-3

If f(x)=-1 has ~~no~~ solutions on the interval [-5, -2], which of the following are possible values for a?

I -4
II -3
III 0

(-4, -4)   (-4, -3)   ~~(-4, 0)~~



Example 3) Use the intermediate value theorem to show that the polynomial has a zero in the interval [0, 1].

$$f(x) = x^3 + 2x - 1$$

$f(0) = -1$

$f(1) = 2$

f is cts b/c it is a polynomial

Since $f(0) < 0 < f(1)$, by the IVT there is a c in [0, 1] such that $f(c) = 0$

Example 4)

Question 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	6	~~4~~	2	~~5~~
2	9	~~2~~	3	~~1~~
3	10	~~−4~~	4	~~2~~
4	−1	~~3~~	6	~~7~~

The functions f and g are ~~differentiable~~ continuous for all real numbers, and g is strictly increasing. The table above gives values of the functions ~~and their first derivatives~~ at selected values of x. The function h is given by $h(x) = f(g(x)) - 6$.

(a) Explain why there must be a value r for $1 < r < 3$ such that $h(r) = -5$.

x

$$h(x) = f(g(x)) - 6$$

$h(1) = f(g(1)) - 6$
$= f(2) - 6$
$= 9 - 6$
$= 3$

$h(3) = f(g(3)) - 6$
$= f(4) - 6$
$= -1 - 6$
$= -7$

h is continuous b/c f+g are cts, and $h(3) < -5 < h(1)$ by the IVT there exists a value r between $[1, 3]$ $1 < r < 3$ such that $h(r) = -5$.

Example 5)

Let f be a function that is continuous on the closed interval $[2, 4]$ with $f(2) = 10$ and $f(4) = 20$. Which of the following is guaranteed by the Intermediate Value Theorem?

(A) $f(x) = 13$ has at least one solution in the open interval $(2, 4)$.

(B) $f(3) = 15$

$f(x) = 13$ on $(2, 4)$

$f(4) = 20$

$f(2) = 10$

Example 6) Verify that the intermediate value theorem applies, then find the value of c guaranteed by the theorem.

$f(x) = x^2 - 6x + 8 \text{ on } [0,3]$ $\qquad f(c) = 0$

$f(0) = 8$

$f(3) = -1$

f is continuous because f is a polynomial, also f(3)<0<f(0) so by the intermediate value theorem there exists a value c in [0, 3] such that f(c)=0.

$0 = x^2 - 6x + 8$

$0 = (x-2)(x-4)$

$\boxed{x = 2}, 4$

Example 7) Verify that the intermediate value theorem applies, then find the value of c guaranteed by the theorem.

$$f(x) = x^2 + x - 1 \ on \ [0,5] \qquad f(c) = 11$$

w

$f(0) = -1$

$f(5) = 29$

f is cts on $[0,5]$ b/c f is a polynomial. Since $f(0) < 11 < f(5)$ by the IVT there exists a value c in $[0,5]$ such that $f(c) = 11$

$f(c) = 11$

$11 = c^2 + c - 1$

$0 = c^2 + c - 12$

$= (c+4)(c-3)$

$c = -4, 3$

$\boxed{c = 3}$

Example 8) Show $e^x = -\ln x$ has a solution.

$$0 = e^x + \ln x$$
$$f(x) = e^x + \ln x$$

The domain of ln is all posive x values, so the funcon is connuous for all x>0. Since e^x is ALWAYS posive, we only need to consider ln(x) which changes from negave values to posive values at x=1.

$$f\left(0^+\right) = e^{0^+} + \ln\left(0^+\right) < 0$$
$$f(1) = e^1 + \ln(1) > 0$$

f is continuous on $(0,\infty)$ and $f(0^+)<0<f(1)$ by the intermediate value theorem there exists a value c in [0,1] such that f(c)=0. Therefore there is a solution to the equation above.